TOWARD BERENSTEIN-ZELEVINSKY DATA IN AFFINE TYPE A, PART II: EXPLICIT DESCRIPTION

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Abstract. In the present paper, we give an explicit description of the affine analogs of Berenstein-Zelevinsky data constructed in [NSS], in terms of certain collections of nonnegative integers, which we call Lusztig data of type $A_{l-1}^{(1)}$.

1. Introduction

1.1. This paper is a continuation of our previous one [NSS], in which we introduced Berenstein-Zelevinsky data of type $A_{l-1}^{(1)}$.

Let us recall the construction of Berenstein-Zelevinsky data of type $A_{l-1}^{(1)}$. We first consider a finite interval I in $\mathbb{Z}$, and the finite-dimensional simple Lie algebra $\mathfrak{g}_I$ of type $A_{|I|}$, where $|I|$ denotes the cardinality of I; here the set I is thought of as the index set of the simple roots of $\mathfrak{g}_I$. Denote by $\mathfrak{h}_I$ the Cartan subalgebra of $\mathfrak{g}_I$, W_I its Weyl group, and by ϖ_i^I, $i \in I$, its fundamental weights. The set Γ_I of chamber weights is defined to be $\Gamma_I := \bigcup_{i\in I} W_I\varpi_i^I \subset \mathfrak{h}_I^*$. Let $\mathbf{M} = (M_\gamma)_{\gamma\in\Gamma_I}$ be a Berenstein-Zelevinsky datum for $\mathfrak{g}_I$ in the sense of Kamnitzer ([Kam1], [Kam2]). It is a collection of integers indexed by the set Γ_I, with some additional conditions called "*edge inequalities*" and "*tropical Plücker relations*" (see Definition 2.3.1). In the general setting of [Kam1] and [Kam2], the set of chamber weights is defined to be a subset of $(\mathfrak{h}_I^\vee)^*$, where $\mathfrak{h}_I^\vee$ is the Cartan subalgebra of the (Langlands) dual Lie algebra $\mathfrak{g}_I^\vee$ of $\mathfrak{g}_I$. However, since we focus on type A, $(\mathfrak{h}_I^\vee)^*$ can be naturally identified with $\mathfrak{h}_I^*$. We denote by $\mathcal{B}\mathcal{Z}_I$ the set of those Berenstein-Zelevinsky data which satisfy the following normalization conditions: $M_{w_0^I\varpi_i^I} = 0$ for all $i \in I$, where w_0^I denotes the longest element of W_I. Kamnitzer showed that $\mathcal{B}\mathcal{Z}_I$ has a crystal structure, under which the $\mathcal{B}\mathcal{Z}_I$ is isomorphic to $B(\infty)$ of type $A_{|I|}$ ([Kam1], [Kam2]).

We note that the family $\{\mathcal{B}\mathcal{Z}_I \mid I \text{ is a finite interval in } \mathbb{Z}\}$ forms a projective system. The set $\mathcal{B}\mathcal{Z}_\mathbb{Z}$ of Berenstein-Zelevinsky data of type A_∞ is defined to be a kind of projective limit of the projective system above. Fix an integer $l \in \mathbb{Z}_{\geq 3}$, and consider a Dynkin diagram automorphism $\sigma : \mathbb{Z} \to \mathbb{Z}$ in type A_∞ given by $\sigma(p) = p + l$. There is a naturally induced action σ on $\mathcal{B}\mathcal{Z}_\mathbb{Z}$. Let $\mathcal{B}\mathcal{Z}_\mathbb{Z}^\sigma$ denote the fixed point subset of $\mathcal{B}\mathcal{Z}_\mathbb{Z}$ under this action of σ. Then, $\mathcal{B}\mathcal{Z}_\mathbb{Z}^\sigma$ has a crystal structure of type $A_{l-1}^{(1)}$, which is induced by the one of $\mathcal{B}\mathcal{Z}_I$. Moreover, there is a particular connected component $\mathcal{B}\mathcal{Z}_\mathbb{Z}^\sigma(\mathbf{O})$ of the crystal $\mathcal{B}\mathcal{Z}_\mathbb{Z}^\sigma$ (see Subsection 4.3), which is isomorphic to $B(\infty)$ of type $A_{l-1}^{(1)}$. Thus, we have a new realization of $B(\infty)$ of type $A_{l-1}^{(1)}$ by affine analogs of Berenstein-Zelevinsky data.

1.2. In this paper, we consider the set of those Berenstein-Zelevinsky data for $\mathfrak{g}_I$ satisfying another normalization conditions: $M_{\varpi_i^I} = 0$ for all $i \in I$; it is denoted by $\mathcal{B}\mathcal{Z}_I^e$. In [S], one of the authors defined a crystal structure on $\mathcal{B}\mathcal{Z}_I^e$, under which the $\mathcal{B}\mathcal{Z}_I^e$ is isomorphic to $B(\infty)$ of type $A_{|I|}$, and constructed an explicit isomorphism $*$ between $\mathcal{B}\mathcal{Z}_I$ and $\mathcal{B}\mathcal{Z}_I^e$.

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As in the case of $\mathcal{B}Z_I$, the family $\{\mathcal{B}Z_I^e \mid I \text{ is a finite interval in } \mathbb{Z}\}$ forms a projective system and we can define $\mathcal{B}Z_{\mathbb{Z}}^e$ as a kind of projective limit of this system. Furthermore, since the map $*$ is compatible with the process of taking this projective limit, we have an isomorphism $*$ between $\mathcal{B}Z_{\mathbb{Z}}$ and $\mathcal{B}Z_{\mathbb{Z}}^e$. Let us denote by $\left(\mathcal{B}Z_{\mathbb{Z}}^e\right)^\sigma$ (*resp.*, $\left(\mathcal{B}Z_{\mathbb{Z}}^e\right)^\sigma(\mathbf{O}^*)$) the image of $\mathcal{B}Z_{\mathbb{Z}}^\sigma$ (*resp.*, $\mathcal{B}Z_{\mathbb{Z}}^\sigma(\mathbf{O})$) under the map $*$. Thus, we obtain a crystal $\left(\mathcal{B}Z_{\mathbb{Z}}^e\right)^\sigma$ of type $A_{l-1}^{(1)}$, and its particular connected component $\left(\mathcal{B}Z_{\mathbb{Z}}^e\right)^\sigma(\mathbf{O}^*)$, which is isomorphic to $B(\infty)$ of type $A_{l-1}^{(1)}$.

1.3. As mentioned in [NSS], we have not yet found an explicit characterization of $\mathcal{B}Z_{\mathbb{Z}}^\sigma(\mathbf{O})$ in terms of "*edge inequalities*" and "*tropical Plücker relations*" in type $A_{l-1}^{(1)}$. Instead, in this paper, we give an explicit description of $\mathcal{B}Z_{\mathbb{Z}}^\sigma(\mathbf{O})$ by another approach.

Let us recall the corresponding results in finite type A. Let $I = [n+1, n+m]$ be a finite interval in $\mathbb{Z}$. In [S], one of the authors gave an explicit description of $\mathcal{B}Z_I^e$ in terms of Lusztig data associated to I. Here a Lusztig datum associated to I is a collection of nonnegative integers indexed by the set $\Delta_I^+ = \{(i,j) \mid n+1 \le i < j \le n+m+1\}$. Denote by $\mathcal{B}_I$ the set of all Lusztig data associated to I. It is known that $\mathcal{B}_I$ parametrizes the canonical basis (or a PBW basis) of $U_q^-(\mathfrak{g}_I)$ ([L1]), and has the induced crystal structure under which the $\mathcal{B}_I$ is isomorphic to $B(\infty)$ of type A_m (see [R], [Sav], and also [S]). Motivated by the work [BFZ] of Berenstein-Fomin-Zelevinsky, one of the authors constructed an explicit isomorphism Φ_I of crystals form $\mathcal{B}_I$ to $\mathcal{B}Z_I^e$.

The notion of Lusztig data can be generalized to the affine case. First, we replace Δ_I^+ by the set $\Delta_{\mathbb{Z}}^+ = \{(i,j) \mid i,j \in \mathbb{Z} \text{ with } i<j\}$, and consider a collection $\mathbf{a} = (a_{i,j})_{(i,j)\in\Delta_{\mathbb{Z}}^+}$ of nonnegative integers indexed by $\Delta_{\mathbb{Z}}^+$ such that $a_{i,j} = 0$ for every $(i,j) \in \Delta_{\mathbb{Z}}^+$ with $j-i \gg 0$, which is called a Lusztig datum associated to $\mathbb{Z}$; we denote by $\mathcal{B}_{\mathbb{Z}}$ the set of all such Lusztig data. Second, we impose the following two conditions on $\mathcal{B}_{\mathbb{Z}}$: for each $(i,j) \in \Delta_{\mathbb{Z}}^+$, $a_{i,j} = a_{i+l,j+l}$ ("cyclicity condition"), and there exits at least one 0 in $\{a_{i+s,j+s} \mid 0 \le s \le l-1\}$ ("aperiodicity condition"). Denote by $\mathcal{B}_{l-1}^{(1),ap}$ the set of those Lusztig data which satisfy the conditions above. Then, this set is naturally identified with the set of aperiodic multisegments over $\mathbb{Z}/l\mathbb{Z}$ in the sense of Lusztig [L3], which parametrizes the canonical basis of the negative part of the quantized universal enveloping algebra of type $A_{l-1}^{(1)}$. The combinatorial description of the induced crystal structure on the latter set is given by Leclerc-Thibon-Vasserot [LTV]. Hence we can endow $\mathcal{B}_{l-1}^{(1),ap}$ with a crystal structure induced by the above identification.

The main result of this paper is as follows (see Theorem 6.4.3 for details). We construct a map $\Phi_{\mathbb{Z}} : \mathcal{B}_{l-1}^{(1),ap} \to \left(\mathcal{B}Z_{\mathbb{Z}}^e\right)^\sigma$ as an affine analog of the map $\Phi_I : \mathcal{B}_I \to \mathcal{B}Z_I^e$, and prove that the image of $\mathcal{B}_{l-1}^{(1),ap}$ under the map $\Phi_{\mathbb{Z}}$ coincides with $\left(\mathcal{B}Z_{\mathbb{Z}}^e\right)^\sigma(\mathbf{O}^*)$. Thus, we obtain an isomorphism $\Phi_{\mathbb{Z}} : \mathcal{B}_{l-1}^{(1),ap} \xrightarrow{\sim} \left(\mathcal{B}Z_{\mathbb{Z}}^e\right)^\sigma(\mathbf{O}^*)$. Also, by composing the map $*$, we obtain an isomorphism $* \circ \Phi_{\mathbb{Z}} : \mathcal{B}_{l-1}^{(1),ap} \xrightarrow{\sim} \mathcal{B}Z_{\mathbb{Z}}^\sigma(\mathbf{O})$. This gives us an explicit description of $\mathcal{B}Z_{\mathbb{Z}}^\sigma(\mathbf{O})$, which we announced in [NSS].

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2. Review on Berenstein-Zelevinsky data associated to finite intervals

2.1. **Root datum of type A_m.** In this subsection, we fix basic notation in type A_m, following [NSS]; however, for our purpose, we need some changes. More specifically, in this paper, we realize the root datum of type A_m as that of $\mathfrak{gl}_{m+1}(\mathbb{C})$, while in [NSS], we realize it as that of $\mathfrak{sl}_{m+1}(\mathbb{C})$.

Let $I = [n+1, n+m]$ be a fixed finite interval in $\mathbb{Z}$. Set $\widetilde{I} := I \cup \{n+m+1\}$ and consider a finite-dimensional complex vector space $\mathfrak{t}_I$ with basis $\{\epsilon_i\}_{i\in\widetilde{I}}$. Let $\mathfrak{t}_I^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}_I, \mathbb{C})$ be the dual space of $\mathfrak{t}_I$. Define $\Lambda_i^I \in \mathfrak{t}_I^*$, $i \in I$, by

$$\langle \epsilon_j, \Lambda_i^I \rangle_I = \begin{cases} 1 & \text{if } j \le i, \\ 0 & \text{if } j > i, \end{cases} \qquad \text{for } i \in I,\ j \in \widetilde{I},$$

where $\langle \cdot, \cdot \rangle_I : \mathfrak{t}_I \times \mathfrak{t}_I^* \to \mathbb{C}$ is the canonical pairing. If we introduce an element κ_I of $\mathfrak{t}_I^*$ by

$$\langle \epsilon_i, \kappa_I \rangle_I = 1 \quad \text{for } i \in \widetilde{I},$$

then $\mathfrak{t}_I^*$ has a basis $\{\Lambda_i^I\}_{i\in I} \cup \{\kappa_I\}$. For $i \in I$, we define

$$h_i := \epsilon_i - \epsilon_{i+1} \in \mathfrak{t}_I \quad \text{and} \quad \alpha_i^I := -\Lambda_{i-1}^I + 2\Lambda_i^I - \Lambda_{i+1}^I \in \mathfrak{t}_I^*.$$

Here we set $\Lambda_{n-1}^I = 0$ and $\Lambda_{n+m+1}^I = \kappa_I$ by convention. Then, $\{\alpha_i^I\}_{i\in I} \cup \{\kappa_I\}$ forms another basis of $\mathfrak{t}_I^*$:

$$\mathfrak{t}_I^* = \Big(\bigoplus_{i\in I} \mathbb{C}\alpha_i^I \Big) \oplus \mathbb{C}\kappa_I. \tag{2.1.1}$$

Also, the matrix $A_I = (a_{ij})_{i,j\in I} := \Big(\langle h_i, \alpha_j^I \rangle_I \Big)_{i,j\in I}$ is the Cartan matrix of type A_m indexed by I. Namely, for $i, j \in I$,

$$a_{ij} = \langle h_j, \alpha_i^I \rangle_I = \begin{cases} 2 & \text{if } j = i, \\ -1 & \text{if } j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{h}_I$ be the subspace of $\mathfrak{t}_I$ (of codimension 1) spanned by the set $\Pi_I^\vee := \{h_i\}_{i\in I}$. Then there is a canonical projection $\pi_I : \mathfrak{t}_I^* \to \mathfrak{h}_I^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_I, \mathbb{C})$, for which $\mathrm{Ker}(\pi_I) = \mathbb{C}\kappa_I$ holds. By the splitting (2.1.1), we can naturally identify $\mathfrak{h}_I^*$ with the subspace $(\mathfrak{t}_I^*)^0$ of $\mathfrak{t}_I^*$ spanned by $\{\alpha_i^I\}_{i\in I}$. Let us denote the induced embedding $\iota_I : \mathfrak{h}_I^* \hookrightarrow \mathfrak{t}_I^*$. Set $\Pi_I := \{\pi_I(\alpha_i^I)\}_{i\in I}$ $(\subset \mathfrak{h}_I^*)$. Thus, $(A_I; \Pi_I, \Pi_I^\vee, \mathfrak{h}_I^*, \mathfrak{h}_I)$ is a root datum of type A_m. Under the above identification, we can write $\Pi_I = \{\alpha_i^I\}_{i\in I} \subset \mathfrak{t}_I^*$.

Let $\mathfrak{g}_I$ be the complex finite-dimensional simple Lie algebra with Cartan matrix A_I and Cartan subalgebra $\mathfrak{h}_I$. Then, $\Pi_I = \{h_i\}_{i\in I}$ and $\Pi_I^\vee = \{\pi_I(\alpha_i)\}_{i\in I}$ are the set of simple coroots and roots of $\mathfrak{g}_I$, respectively. Also, $\varpi_i^I := \pi_I(\Lambda_i^I) \in \mathfrak{h}_I^*$, $i \in I$, are the fundamental weights for $\mathfrak{g}_I$.

We define a linear automorphism σ_i, $i \in I$, of $\mathfrak{t}_I^*$ by

$$\sigma_i(\lambda) := \lambda - \langle h_i, \lambda \rangle_I \alpha_i^I \quad \text{for } \lambda \in \mathfrak{t}_I^*,$$

and the linear automorphism s_i, $i \in I$ of $\mathfrak{h}_I^*$ by

$$s_i := \pi_I \circ \sigma_i \circ \iota_I.$$

Then we have

$$s_i(\mu) = \mu - \langle h_i, \mu \rangle \pi_I(\alpha_i^I) \quad \text{for } \mu \in \mathfrak{h}_I^*,$$

where $\langle \cdot, \cdot \rangle : \mathfrak{h}_I \times \mathfrak{h}_I^* \to \mathbb{C}$ is the canonical pairing. The group $W_{\mathfrak{h}_I^*} := \langle s_i \mid i \in I \rangle$ $\big(\subset \mathrm{Aut}(\mathfrak{h}_I^*)\big)$ is thought of as the Weyl group of $\mathfrak{g}_I$. Because $W_{\mathfrak{h}_I^*}$ is isomorphic to $W_{\mathfrak{t}_I^*} :=$

$\langle \sigma_i \mid i \in I \rangle$ $\bigl(\subset \mathrm{Aut}(\mathfrak{t}_I^*)\bigr)$ via the map $s_i \mapsto \sigma_i$, we write $W_I = W_{\mathfrak{t}_I^*}$ for the remainder of this paper.

2.2. **Chamber weights.** Consider the set $\{w\varpi_i^I \mid w \in W_{\mathfrak{h}_I^*},\ i \in I\}$ $\bigl(\subset \mathfrak{h}_I^*\bigr)$. In [NSS], this set is denoted by Γ_I, and an element of it is called a chamber weight. However, in this paper, we change the notation. Namely, we set

$$\Gamma_I := \{w\Lambda_i^I \mid w \in W_I,\ i \in I\} \ \bigl(\subset \mathfrak{t}_I^*\bigr),$$

and call an element of this set a chamber weight. Since there is a bijection $\Gamma_I \xrightarrow{\sim} \{w\varpi_i^I \mid w \in W_{\mathfrak{h}_I^*},\ i \in I\}$ induced by $\pi_I : \mathfrak{t}_I^* \to \mathfrak{h}_I^*$, these two sets are naturally identified. Thus, the set of chamber weights in the sense of [NSS] is equal to $\pi_I(\Gamma_I)$.

Let w_0^I denote the longest element of $W_{\mathfrak{h}_I^*}$, and $\omega_I : I \to I$ an involution defined by $n+i \mapsto n+m-i+1$, $1 \le i \le m$. As in [NSS], the following two equalities are verified in $\mathfrak{h}_I^*$ and in $W_{\mathfrak{h}_I^*}$, respectively: for $i \in I$,

$$w_0^I(\varpi_i^I) = -\varpi_{\omega_I(i)}^I, \quad w_0^I s_{\omega_I(i)} = s_i w_0^I. \tag{2.2.1}$$

Let $\gamma' \in \pi_I(\Gamma_I)$. Recall that it can be written as $\gamma' = w\varpi_i^I$ for some $w \in W_{\mathfrak{h}_I^*}$ and $i \in I$. Because of (2.2.1), we have

$$-\gamma' = ww_0^I\varpi_{\omega_I(i)} \tag{2.2.2}$$

and

$$\{-w\varpi_i^I \mid w \in W_{\mathfrak{h}_I^*},\ i \in I\} = \{w\varpi_i^I \mid w \in W_{\mathfrak{h}_I^*},\ i \in I\}.$$

Now we consider analogs of (2.2.1) for $\mathfrak{t}_I^*$. As in the case above, we have

$$w_0^I\sigma_i = \sigma_{\omega_I(i)}w_0^I, \tag{2.2.3}$$

where we also denote by w_0^I the longest element of $W_I = W_{\mathfrak{t}_I^*}$. However, an obvious analog of the first equality of (2.2.1) fails. In fact, we have

$$w_0^I(\Lambda_i^I) = \kappa_I - \Lambda_{\omega_I(i)}^I.$$

As a consequence, Γ_I does not coincide with $-\Gamma_I$. Still, we can define an analog of the map $\gamma' \mapsto -\gamma'$ as follows. For $\gamma = w\Lambda_i^I$ with $w \in W_I, i \in I$, we set

$$\gamma^c := ww_0^I\Lambda_{\omega_I(i)}.$$

Then, it is easy to see that the map $c : \gamma \mapsto \gamma^c$ is well-defined, and gives an involution of Γ_I. In this paper, this map plays an important role.

A set of integers $\mathbf{k} = \{k_1 < k_2 < \cdots < k_u\} \subset \widetilde{I}$ is called a Maya diagram associated to I. We denote by $\mathcal{M}_I$ the set of all Maya diagrams associated to I, and set $\mathcal{M}_I^\times := \mathcal{M}_I \setminus \{\phi, \widetilde{I}\}$. For a given chamber weight $\gamma \in \Gamma_I$, we define the associated Maya diagram $\mathbf{k}(\gamma)$ by

$$\mathbf{k}(\gamma) := \{k \in \widetilde{I} \mid \langle \epsilon_k, \gamma \rangle_I = 1\}.$$

Note that $\langle \epsilon_k, \gamma \rangle_I = 0$ or 1 for each $k \in \widetilde{I}$ and $\gamma \in \Gamma_I$ by the definitions. Since the map $\gamma \mapsto \mathbf{k}(\gamma)$ gives a bijection from Γ_I to $\mathcal{M}_I^\times$, we can identify these two sets.

Under the identification above, the involution c is described as follows (see [S]). For a Maya diagram $\mathbf{k} \in \mathcal{M}_I^\times$, define $\mathbf{k}^c := \widetilde{I} \setminus \mathbf{k}$. Then the next lemma follows easily.

Lemma 2.2.1.

$$\mathbf{k}(\gamma^c) = \mathbf{k}(\gamma)^c \quad (\gamma \in \Gamma_I).$$

2.3. **BZ data associated to I.** Let $\mathbf{M}=(M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_I^{\times}}$ be a collection of integers indexed by $\mathcal{M}_I^{\times}$. For each $\mathbf{k}\in\mathcal{M}_I^{\times}$, we call $M_{\mathbf{k}}$ the $\mathbf{k}$-component of $\mathbf{M}$ and denote it by $(\mathbf{M})_{\mathbf{k}}$. Under the identification $\Gamma_I\cong\mathcal{M}_I^{\times}$, Berenstein-Zelevinsky data are defined as follows (see [S]).

Definition 2.3.1. *A collection $\mathbf{M}=(M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_I^{\times}}$ of integers indexed by $\mathcal{M}_I^{\times}$ is called a Berenstein-Zelevinsky (BZ for short) datum associated to an interval I if it satisfies the following conditions:*

(BZ-1) *for all indices $i\neq j$ in $\widetilde{I}$ and all $\mathbf{k}\in\mathcal{M}_n$ with $\mathbf{k}\cap\{i,j\}=\phi$,*

$$M_{\mathbf{k}\cup\{i\}}+M_{\mathbf{k}\cup\{j\}}\leq M_{\mathbf{k}\cup\{i,j\}}+M_{\mathbf{k}};$$

(BZ-2) *for all indices $i<j<k$ in $\widetilde{I}$ and all $\mathbf{k}\in\mathcal{M}_n$ with $\mathbf{k}\cap\{i,j,k\}=\phi$,*

$$M_{\mathbf{k}\cup\{i,k\}}+M_{\mathbf{k}\cup\{j\}}=\min\left\{M_{\mathbf{k}\cup\{i,j\}}+M_{\mathbf{k}\cup\{k\}},\ M_{\mathbf{k}\cup\{j,k\}}+M_{\mathbf{k}\cup\{i\}}\right\}.$$

Here we set $M_{\phi}=M_{\widetilde{I}}=0$ by convention.

Definition 2.3.2. *A BZ datum $\mathbf{M}=(M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_I^{\times}}$ is called a w_0-BZ (resp., e-BZ) datum if it satisfies the following normalization conditions:*

(BZ-0) *for every $i\in I$,*

$$M_{\mathbf{k}(\Lambda_i^I)^c}=0 \qquad (\textit{resp.},\ M_{\mathbf{k}(\Lambda_i^I)}=0).$$

We denote by $\mathcal{BZ}_I$ (*resp.*, $\mathcal{BZ}_I^e$) the set of all w_0-BZ (*resp.*, e-BZ) data. Consider a map $*:\mathcal{BZ}_I\to\mathcal{BZ}_I^e$ by

$$(\mathbf{M}^*)_{\mathbf{k}}:=(\mathbf{M})_{\mathbf{k}^c}\quad(\mathbf{M}\in\mathcal{BZ}_I).$$

By [S], this is a well-defined bijection; the inverse of it is also denoted by $*$.

Let us discuss crystal structures on BZ data. First, we define a crystal structure on $\mathcal{BZ}_I$, following Kamnitzer [Kam2] (see also [S]). For $\mathbf{M}=(M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_I^{\times}}\in\mathcal{BZ}_I$, we define the weight $\mathrm{wt}(\mathbf{M})$ of $\mathbf{M}$ by

$$\mathrm{wt}(\mathbf{M}):=\sum_{i\in I}M_{\mathbf{k}(\Lambda_i^I)}\alpha_i^I.$$

For $i\in I$, we set

$$\varepsilon_i(\mathbf{M}):=-\left(M_{\mathbf{k}(\Lambda_i^I)}+M_{\mathbf{k}(\sigma_i\Lambda_i^I)}-M_{\mathbf{k}(\Lambda_{i-1}^I)}-M_{\mathbf{k}(\Lambda_{i+1}^I)}\right),$$

$$\varphi_i(\mathbf{M}):=\varepsilon_i(\mathbf{M})+\langle h_i,\mathrm{wt}(\mathbf{M})\rangle_I.$$

In order to define the action of Kashiwara operators $\widetilde{e}_i$ and $\widetilde{f}_i$, $i\in I$, we recall the following result, due to Kamnitzer:

Proposition 2.3.3 ([Kam2])**.** *Let $\mathbf{M}=(M_{\mathbf{k}})\in\mathcal{BZ}_I$ be a w_0-BZ datum associated to I.*

(1) *If $\varepsilon_i(\mathbf{M})>0$, then there exists a unique w_0-BZ datum, denoted by $\widetilde{e}_i\mathbf{M}$, such that*
- (i) $(\widetilde{e}_i\mathbf{M})_{\mathbf{k}(\Lambda_i^I)}=M_{\mathbf{k}(\Lambda_i^I)}+1$,
- (ii) $(\widetilde{e}_i\mathbf{M})_{\mathbf{k}}=M_{\mathbf{k}}$ *for all* $\mathbf{k}\in\mathcal{M}_I^{\times}\setminus\mathcal{M}_I^{\times}(i)$.

Here, $\mathcal{M}_I^{\times}(i)=\left\{\mathbf{k}\in\mathcal{M}_I^{\times}\mid i\in\mathbf{k}\text{ and }i+1\notin\mathbf{k}\right\}\subset\mathcal{M}_I^{\times}$.

(2) *There exists a unique w_0-BZ datum, denoted by $\widetilde{f}_i\mathbf{M}$, such that*
- (iii) $(\widetilde{f}_i\mathbf{M})_{\mathbf{k}(\Lambda_i^I)}=M_{\mathbf{k}(\Lambda_i^I)}-1$,
- (iv) $(\widetilde{f}_i\mathbf{M})_{\mathbf{k}}=M_{\mathbf{k}}$ *for all* $\mathbf{k}\in\mathcal{M}_n^{\times}\setminus\mathcal{M}_n^{\times}(i)$.

If $\varepsilon_i(\mathbf{M})=0$, then we set $\widetilde{e}_i\mathbf{M}=0$.

Theorem 2.3.4 ([Kam2]). *The set $\mathcal{BZ}_I$, equipped with the maps* $\mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i$, *is a crystal which is isomorphic to* $\left(B(\infty); \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right)$.

Second, we introduce a crystal structure on $\mathcal{BZ}_I^e$. For $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_I^\times} \in \mathcal{BZ}_I^e$, we set

$$\mathrm{wt}(\mathbf{M}) := \mathrm{wt}(\mathbf{M}^*), \quad \varepsilon_i^*(\mathbf{M}) := \varepsilon_i(\mathbf{M}^*), \quad \varphi_i^*(\mathbf{M}) := \varphi_i(\mathbf{M}^*).$$

Corollary 2.3.5 ([S]). (1) *Let $\mathbf{M} = (M_{\mathbf{k}}) \in \mathcal{BZ}_I^e$ be an e-BZ datum associated to I. If $\varepsilon_i^*(\mathbf{M}) > 0$, then there exists a unique e-BZ datum, denoted by $\widetilde{e}_i^*\mathbf{M}$, such that*

(i) $(\widetilde{e}_i^*\mathbf{M})_{\mathbf{k}(\Lambda_i^I)^c} = M_{\mathbf{k}(\Lambda_i^I)^c} + 1$,

(ii) $(\widetilde{e}_i^*\mathbf{M})_{\mathbf{k}} = M_{\mathbf{k}}$ *for all* $\mathbf{k} \in \mathcal{M}_I^\times \setminus \mathcal{M}_I^\times(i)^*$.

Here, $\mathcal{M}_I^\times(i)^* = \left\{\mathbf{k} \in \mathcal{M}_I^\times \mid i \notin \mathbf{k} \text{ and } i+1 \in \mathbf{k}\right\} \subset \mathcal{M}_I^\times$.

(2) *There exists a unique e-BZ datum, denoted by $\widetilde{f}_i^*\mathbf{M}$, such that*

(iii) $(\widetilde{f}_i^*\mathbf{M})_{\mathbf{k}(\Lambda_i^I)^c} = M_{\mathbf{k}(\Lambda_i^I)^c} - 1$,

(iv) $(\widetilde{f}_i^*\mathbf{M})_{\mathbf{k}} = M_{\mathbf{k}}$ *for all* $\mathbf{k} \in \mathcal{M}_I^\times \setminus \mathcal{M}_I^\times(i)^*$.

(3) *If $\varepsilon_i^*(\mathbf{M}) = 0$, then we set $\widetilde{e}_i^*\mathbf{M} := 0$. The following equalities hold:*

$$\widetilde{e}_i^*\mathbf{M} = (\widetilde{e}_i(\mathbf{M}^*))^*, \quad \widetilde{f}_i^*\mathbf{M} = (\widetilde{f}_i(\mathbf{M}^*))^*.$$

Here it is understood that $0^* = 0$.

(4) *The set $\mathcal{BZ}_I^e$, equipped with the maps* $\mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*$, *is a crystal which is isomorphic to* $\left(B(\infty); \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right)$. *Moreover, the bijection $* : \mathcal{BZ}_I \xrightarrow{\sim} \mathcal{BZ}_I^e$ gives an isomorphism of crystals form* $\left(\mathcal{BZ}_I; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right)$ *to* $\left(\mathcal{BZ}_I^e; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right)$.

2.4. **Lusztig data associated to I.** Recall that $I = [n+1, n+m]$. Let $\Delta_I^+ = \{(i,j) \mid i,j \in \widetilde{I} \text{ with } i<j\}$, and consider

$$\mathcal{B}_I := \left\{\mathbf{a} = (a_{i,j})_{(i,j)\in\Delta_I^+} \mid a_{i,j} \in \mathbb{Z}_{\geq 0} \text{ for all } (i,j) \in \Delta_I^+\right\},$$

which is the set of all $m(m+1)/2$-tuples of nonnegative integers indexed by Δ_I^+. In this paper, an element of $\mathcal{B}_I$ is called a Lusztig datum associated to I.

We define two crystal structures on $\mathcal{B}_I$, following [S]. For $\mathbf{a} \in \mathcal{B}_I$, define the weight $\mathrm{wt}(\mathbf{a})$ of $\mathbf{a}$ by

$$\mathrm{wt}(\mathbf{a}) := -\sum_{i\in I} r_i(\mathbf{a})\alpha_i^I, \quad \text{where} \quad r_i(\mathbf{a}) := \sum_{s=n+1}^{i}\sum_{t=i+1}^{n+m+1} a_{s,t} \quad (i \in I).$$

For $i \in I$, set

$$A_k^{(i)}(\mathbf{a}) := \sum_{s=n+1}^{k}(a_{s,i+1} - a_{s-1,i}) \quad (n+1 \leq k \leq i),$$

$$A_k^{*(i)}(\mathbf{a}) := \sum_{t=k+1}^{n+m+1}(a_{i,t} - a_{i+1,t+1}) \quad (i \leq k \leq n+m).$$

Here we set $a_{n,i} = 0$ and $a_{i+1,n+m+2} = 0$ by convention. Next, define

$$\varepsilon_i(\mathbf{a}) := \max\left\{A_{n+1}^{(i)}(\mathbf{a}), \ldots, A_i^{(i)}(\mathbf{a})\right\}, \quad \varphi_i(\mathbf{a}) = \varepsilon_i(\mathbf{a}) + \langle h_i, \mathrm{wt}(\mathbf{a})\rangle,$$

$$\varepsilon_i^*(\mathbf{a}) := \max\left\{A_i^{*(i)}(\mathbf{a}), \ldots, A_{n+m}^{*(i)}(\mathbf{a})\right\}, \quad \varphi_i^*(\mathbf{a}) = \varepsilon_i^*(\mathbf{a}) + \langle h_i, \mathrm{wt}(\mathbf{a})\rangle.$$

We set

$$\begin{aligned} k_e &:= \min\left\{ n+1 \le k \le i \;\middle|\; \varepsilon_i(\mathbf{a}) = A_k^{(i)}(\mathbf{a}) \right\}, \\ k_f &:= \max\left\{ n+1 \le k \le i \;\middle|\; \varepsilon_i(\mathbf{a}) = A_k^{(i)}(\mathbf{a}) \right\}, \\ k_e^* &:= \max\left\{ i \le k \le n+m \;\middle|\; \varepsilon_i^*(\mathbf{a}) = A_k^{*(i)}(\mathbf{a}) \right\}, \\ k_f^* &:= \min\left\{ i \le k \le n+m \;\middle|\; \varepsilon_i^*(\mathbf{a}) = A_k^{*(i)}(\mathbf{a}) \right\}. \end{aligned}$$

For a given $\mathbf{a} \in \mathcal{B}_I$, we introduce the following $m(m+1)/2$-tuples of integers $\mathbf{a}^{(\sharp)} = \left(a_{s,t}^{(\sharp)}\right)$ $(\sharp = 1,2,3,4)$ by

$$a_{s,t}^{(1)} := \begin{cases} a_{k_e,i}+1 & \text{if } s=k_e,\ t=i, \\ a_{k_e,i+1}-1 & \text{if } s=k_e,\ t=i+1, \\ a_{s,t} & \text{otherwise.} \end{cases}$$

$$a_{s,t}^{(2)} := \begin{cases} a_{k_f,i}-1 & \text{if } s=k_f,\ t=i, \\ a_{k_f,i+1}+1 & \text{if } s=k_f,\ t=i+1, \\ a_{s,t} & \text{otherwise,} \end{cases}$$

$$a_{s,t}^{(3)} := \begin{cases} a_{i,k_e^*+1}-1 & \text{if } s=i,\ t=k_e^*+1, \\ a_{i+1,k_e^*+1}+1 & \text{if } s=i+1,\ t=k_e^*+1, \\ a_{s,t} & \text{otherwise.} \end{cases}$$

$$a_{s,t}^{(4)} := \begin{cases} a_{i,k_f^*+1}+1 & \text{if } s=i,\ t=k_f^*+1, \\ a_{i+1,k_f^*+1}-1 & \text{if } s=i+1,\ t=k_f^*+1, \\ a_{s,t} & \text{otherwise.} \end{cases}$$

Finally, we define Kashiwara operators on $\mathcal{B}_I$ by:

$$\widetilde{e}_i\mathbf{a} = \begin{cases} \mathbf{0} & \text{if } \varepsilon_i(\mathbf{a}) = 0, \\ \mathbf{a}^{(1)} & \text{if } \varepsilon_i(\mathbf{a}) > 0, \end{cases} \qquad \widetilde{f}_i\mathbf{a} = \mathbf{a}^{(2)},$$

$$\widetilde{e}_i^*\mathbf{a} = \begin{cases} \mathbf{0} & \text{if } \varepsilon_i^*(\mathbf{a}) = 0, \\ \mathbf{a}^{(3)} & \text{if } \varepsilon_i^*(\mathbf{a}) > 0, \end{cases} \qquad \widetilde{f}_i^*\mathbf{a} = \mathbf{a}^{(4)}.$$

Proposition 2.4.1 ([R],[S]). *Each of* $\left(\mathcal{B}_I; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right)$ *and* $\left(\mathcal{B}_I; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right)$ *is a crystal which is isomorphic to* $B(\infty)$.

This proposition tells us that $\mathcal{B}_I$ has two different crystal structures, which are isomorphic to each other. In this paper, we call the first one the ordinary crystal structure on $\mathcal{B}_I$; the second one is called the $*$-crystal structure on $\mathcal{B}_I$.

Definition 2.4.2 ([BFZ]). *Let* $\mathbf{k} = \{k_{n+1} < k_{n+2} < \cdots < k_{n+u}\} \in \mathcal{M}_I^{\times}$. *For such a* $\mathbf{k}$, *we define a* $\mathbf{k}$*-tableau as an uppertriangular matrix* $C = (c_{p,q})_{n+1 \le p \le q \le n+u}$ *with integer entries satisfying*

$$c_{p,p} = k_p \qquad (n+1 \le p \le n+u),$$

and the (usual) monotonicity conditions for semi-standard tableaux:

$$c_{p,q} \le c_{p,q+1}, \qquad c_{p,q} < c_{p+1,q}.$$

For a given Lusztig datum $\mathbf{a} = (a_{i,j}) \in \mathcal{B}_I$, let $\mathbf{M}(\mathbf{a}) = (M_{\mathbf{k}}(\mathbf{a}))_{\mathbf{k} \in \mathcal{M}_I^{\times}}$ be a collection of integers defined by

$$M_{\mathbf{k}}(\mathbf{a}) := -\sum_{j=n+1}^{n+u} \sum_{i=n+1}^{k_j-1} a_{i,k_j} + \min\left\{ \sum_{n+1 \le p < q \le n+u} a_{c_{p,q},c_{p,q}+(q-p)} \;\middle|\; \begin{array}{c} C = (c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau} \end{array} \right\}. \tag{2.4.1}$$

We denote the map $\mathbf{a} \mapsto \mathbf{M}(\mathbf{a})$ by Φ_I.

Theorem 2.4.3 ([BFZ], [S]). *For each $\mathbf{a} \in \mathcal{B}_I$, $\Phi_I(\mathbf{a}) = \mathbf{M}(\mathbf{a})$ is an e-BZ datum. Moreover, the map $\Phi_I : \mathcal{B}_I \to \mathcal{BZ}_I^e$ is a bijection, which induces an isomorphism of crystals*

$$\left(\mathcal{B}_I; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right) \xrightarrow{\Phi_I} \left(\mathcal{B}Z_I^e; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right).$$

Consider a composition of bijections $\mathcal{B}_I \xrightarrow{\Phi_I} \mathcal{B}Z_I^e \xrightarrow{*} \mathcal{B}Z_I$. By Corollary 2.3.5 and Theorem 2.4.3, we obtain the following corollary.

Corollary 2.4.4. *The bijection $* \circ \Phi_I : \mathcal{B}_I \to \mathcal{B}Z_I$ gives an isomorphism of crystals*

$$\left(\mathcal{B}_I; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right) \xrightarrow{*\circ\Phi_I} \left(\mathcal{B}Z_I; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right).$$

3. Berenstein-Zelevinsky data associated to $\mathbb{Z}$

3.1. **Root datum of $\mathfrak{gl}_\infty(\mathbb{C})$.** In this subsection, we introduce basic notation in type A_∞, following [NSS]; however, for our purpose, we need some changes as in the finite interval case. More specifically, we need to replace $\mathfrak{sl}_\infty(\mathbb{C})$ by $\mathfrak{gl}_\infty(\mathbb{C})$.

Let $\mathfrak{t}$ be a vector space over $\mathbb{C}$ with basis $\{\epsilon_i\}_{i\in\mathbb{Z}}$. Set

$$h_i := \epsilon_i - \epsilon_{i+1} \quad \text{for } i \in \mathbb{Z},$$

and consider the subspace $\mathfrak{h}$ of $\mathfrak{t}$ spanned by $\Pi = \{h_i\}_{i\in\mathbb{Z}}$. Note that $\mathfrak{h}$ is a subspace of $\mathfrak{t}$ of codimension 1.

Let $\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ (*resp.*, $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$) be the dual space of $\mathfrak{t}$ (*resp.*, $\mathfrak{h}$). Then there is a canonical projection $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$, whose kernel is spanned by an element κ of $\mathfrak{t}^*$ defined by $\kappa(\epsilon_i) = 1$ for all $i \in \mathbb{Z}$. We remark that there is a splitting of $\mathfrak{t}^*$:

$$\mathfrak{t}^* \cong \mathfrak{h}^* \oplus \mathbb{C}\kappa. \tag{3.1.1}$$

Define Λ_i, $\Lambda_i^c \in \mathfrak{t}^*$, $i \in \mathbb{Z}$, by

$$\langle \epsilon_j, \Lambda_i \rangle_{\mathbb{Z}} := \begin{cases} 1 & \text{if } j \le i, \\ 0 & \text{if } j > i, \end{cases} \qquad \langle \epsilon_j, \Lambda_i^c \rangle_{\mathbb{Z}} := \begin{cases} 0 & \text{if } j \le i, \\ 1 & \text{if } j > i, \end{cases}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{Z}} : \mathfrak{t} \times \mathfrak{t}^* \to \mathbb{C}$ is the canonical pairing. Then we have

$$\langle h_j, \Lambda_i \rangle_{\mathbb{Z}} = \delta_{ij},\ \langle \epsilon_0, \Lambda_i \rangle_{\mathbb{Z}} = \begin{cases} 1 & \text{if } i \ge 0, \\ 0 & \text{if } i < 0, \end{cases} \quad \langle h_j, \Lambda_i^c \rangle_{\mathbb{Z}} = -\delta_{ij},\ \langle \epsilon_0, \Lambda_i^c \rangle_{\mathbb{Z}} = \begin{cases} 0 & \text{if } i \ge 0, \\ 1 & \text{if } i < 0. \end{cases}$$

These formulas enable us to deduce the following lemma:

Lemma 3.1.1. *For each $i \in \mathbb{Z}$, we have $\pi(\Lambda_i^c) = -\pi(\Lambda_i)$.*

Set

$$\alpha_i := -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1} \quad \text{for } i \in \mathbb{Z}.$$

By the definitions, we have

$$\langle h_j, \alpha_i \rangle_{\mathbb{Z}} = \begin{cases} 2 & \text{if } j = i, \\ -1 & \text{if } j = i \pm 1, \\ 0 & \text{otherwise}, \end{cases} \qquad \langle \epsilon_j, \alpha_i \rangle_{\mathbb{Z}} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = j - 1, \\ 0 & \text{otherwise}. \end{cases}$$

Let $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ be the canonical pairing. Then, $\langle h_j, \pi(\alpha_i) \rangle = \langle h_j, \alpha_i \rangle_{\mathbb{Z}}$, and $\Pi^\vee := \{\pi(\alpha_i)\}_{i\in\mathbb{Z}}$ is a linearly independent subset of $\mathfrak{h}^*$. In other words, $(A_{\mathbb{Z}}, \Pi, \Pi^\vee, \mathfrak{h}^*, \mathfrak{h})$ is a root datum of type A_∞ in the sense of [NSS]. Here, $A_{\mathbb{Z}} = (a_{ij})_{i,j\in\mathbb{Z}}$ is the Cartan matrix of type A_∞, whose entries a_{ij} are defined by $a_{ij} := \langle h_j, \pi(\alpha_i) \rangle$ for $i, j \in \mathbb{Z}$.

For each $i \in \mathbb{Z}$, Define $\sigma_i \in \mathrm{Aut}(\mathfrak{t}^*)$ by

$$\sigma_i(\lambda) := \lambda - \langle h_i, \lambda \rangle_{\mathbb{Z}} \alpha_i \quad \text{for } \lambda \in \mathfrak{t}^*.$$

Similarly, define $\sigma_i \in \mathrm{Aut}(\mathfrak{t})$ by

$$\sigma_i(t) := t - \langle t, \alpha_i \rangle_{\mathbb{Z}} h_i \quad \text{for } t \in \mathfrak{t}.$$

Let $\iota : \mathfrak{h}^* \hookrightarrow \mathfrak{t}^*$ be the embedding induced by the splitting (3.1.1). If we define $s_i \in \mathrm{Aut}(\mathfrak{h}^*)$ by

$$s_i := \pi \circ \sigma_i \circ \iota,$$

then we have

$$s_i(\mu) = \mu - \langle h_i, \mu \rangle \pi(\alpha_i) \quad \text{for } \mu \in \mathfrak{h}^*.$$

Let $W_{\mathbb{Z}} := \langle s_i \mid i \in \mathbb{Z} \rangle \subset \mathrm{Aut}(\mathfrak{h}^*)$ be the Weyl group of type A_∞. Consider the subgroup of $\mathrm{Aut}(\mathfrak{t}^*)$ generated by σ_i, $i \in \mathbb{Z}$. It is easy to see that this group is isomorphic to $W_{\mathbb{Z}}$. For this reason, we also denote it by $W_{\mathbb{Z}}$.

3.2. **Chamber weights associated to $\mathbb{Z}$ and Maya diagrams.** Set

$$\Xi_{\mathbb{Z}} = \{ w\Lambda_i \mid w \in W_{\mathbb{Z}},\ i \in \mathbb{Z} \} \quad \text{and} \quad \Gamma_{\mathbb{Z}} = \{ w\Lambda_i^c \mid w \in W_{\mathbb{Z}},\ i \in \mathbb{Z} \}.$$

An element of $\Xi_{\mathbb{Z}}$ (*resp.*, $\Gamma_{\mathbb{Z}}$) is called a chamber weight (*resp.*, dual chamber weight) associated to $\mathbb{Z}$.

Definition 3.2.1. (1) *For a given integer $r \in \mathbb{Z}$, a subset $\mathbf{k}$ of $\mathbb{Z}$ is called a Maya diagram of charge r if it satisfies the following condition: there exist nonnegative integers p and q such that*

$$\mathbb{Z}_{\le r-p} \subset \mathbf{k} \subset \mathbb{Z}_{\le r+q}, \quad |\mathbf{k} \cap \mathbb{Z}_{>r-p}| = p, \tag{3.2.1}$$

where $|\mathbf{k} \cap \mathbb{Z}_{>r-p}|$ denotes the cardinality of the finite set $\mathbf{k} \cap \mathbb{Z}_{>r-p}$. We denote by $\mathcal{M}_{\mathbb{Z}}^{(r)}$ the set of all Maya diagrams of charge r, and set $\mathcal{M}_{\mathbb{Z}} := \bigcup_{r \in \mathbb{Z}} \mathcal{M}_{\mathbb{Z}}^{(r)}$. In particular, the element $\mathbb{Z}_{\le r} \in \mathcal{M}_{\mathbb{Z}}^{(r)}$ is called the ground state of charge r.
(2) *For a Maya diagram $\mathbf{k}$ of charge r, let $\mathbf{k}^c := \mathbb{Z} \setminus \mathbf{k}$ be the complement of $\mathbf{k}$ in $\mathbb{Z}$. We call $\mathbf{k}^c$ the complementary Maya diagram of charge r associated to $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{(r)}$. We denote by $\mathcal{M}_{\mathbb{Z}}^{(r),c}$ the set of all complementary Maya diagrams of charge r, and set $\mathcal{M}_{\mathbb{Z}}^c := \bigcup_{r \in \mathbb{Z}} \mathcal{M}_{\mathbb{Z}}^{(r),c}$. In particular, the element $(\mathbb{Z}_{\le r})^c = \mathbb{Z}_{>r} \in \mathcal{M}_{\mathbb{Z}}^{(r),c}$ is called the complementary ground state of charge r.*

From the definition above, a Maya diagram $\mathbf{k}$ of charge r can be regarded as a sequence of integers indexed by $\mathbb{Z}_{\le r}$:

$$\mathbf{k} = \{ k_j \mid j \in \mathbb{Z}_{\le r} \} \text{ such that } k_{j-1} < k_j \text{ for } j \le r,\ k_j = j \text{ for } j \ll r.$$

Similarly, the complementary Maya diagram $\mathbf{k}$ of charge r can be regarded as a sequence of integers indexed by $\mathbb{Z}_{>r}$:

$$\mathbf{k} = \{ k_j \mid j \in \mathbb{Z}_{>r} \} \text{ such that } k_j < k_{j+1} \text{ for } j > r,\ k_j = j \text{ for } j \gg r.$$

The map $c : \mathcal{M}_{\mathbb{Z}} \to \mathcal{M}_{\mathbb{Z}}^c$ defined by $\mathbf{k} \mapsto \mathbf{k}^c$ is a bijection; the inverse of this map is also denoted by c.

Fix an integer $r \in \mathbb{Z}$. It is well-known that $\mathcal{M}_{\mathbb{Z}}^{(r)}$ can be identified with the set $\mathcal{Y}$ of all Young diagrams in the following way. Let $\mathbf{k} = \{ k_j \mid j \in \mathbb{Z}_{\le r} \} \in \mathcal{M}_{\mathbb{Z}}^{(r)}$ be a Maya diagram of charge r, and $p, q \in \mathbb{Z}$ integers satisfying the condition (3.2.1). Define an increasing sequence of nonnegative integers $0 \le m_0 < m_1 < \cdots < m_{p-1} \le p+q-1$ by

$$\mathbf{k} \cap \mathbb{Z}_{>r-p} := \{ m_0 + (r-p+1),\ m_1 + (r-p+1),\ \ldots,\ m_{p-1} + (r-p+1) \}.$$

Notice that $0 \le m_0 \le m_1 - 1 \le \cdots \le m_{p-1} - (p-1) \le q$. Hence we can define a Young diagram $Y(\mathbf{k}) \in \mathcal{Y}$ by

$$Y(\mathbf{k}) := (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p \ge 0),$$

where $\lambda_j := m_{p-j} - (p-j)$ for $1 \le j \le p$. It is easily checked that the definition above of $Y(\mathbf{k})$ does not depend on the choice of integers p, q satisfying the condition (3.2.1), and that the map $\mathbf{k} \mapsto Y(\mathbf{k})$ gives a bijection from $\mathcal{M}_{\mathbb{Z}}^{(r)}$ to $\mathcal{Y}$. We remark that, for $\mathbb{Z}_{\le r} \in \mathcal{M}_{\mathbb{Z}}^{(r)}$, the corresponding Young diagram $Y(\mathbb{Z}_{\le r})$ is the empty set ϕ.

For $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}$, we set $Y(\mathbf{k}) := Y(\mathbf{k}^c)$. The map $\mathbf{k} \mapsto Y(\mathbf{k})$ also gives a bijection from $\mathcal{M}_{\mathbb{Z}}^{(r),c}$ to $\mathcal{Y}$.

We identify $\Xi_{\mathbb{Z}}$ (*resp.*, $\Gamma_{\mathbb{Z}}$) with $\mathcal{M}_{\mathbb{Z}}$ (*resp.*, $\mathcal{M}_{\mathbb{Z}}^c$); we only give an explicit identification of $\Xi_{\mathbb{Z}}$ and $\mathcal{M}_{\mathbb{Z}}$, since we can identify $\Gamma_{\mathbb{Z}}$ with $\mathcal{M}_{\mathbb{Z}}^c$ in a similar way.

Recall that, for $\xi \in \Xi_{\mathbb{Z}}$, there uniquely exists $r \in \mathbb{Z}$ such that $\xi = w\Lambda_r$ for $w \in W_{\mathbb{Z}}$. By the definitions, we have $\langle \epsilon_k, \xi \rangle_{\mathbb{Z}} = 0$ or 1 for each $k \in \mathbb{Z}$. For a given $\xi \in \Xi_{\mathbb{Z}}$, define a subset of $\mathbb{Z}$ by

$$\mathbf{k}(\xi) := \{k \in \mathbb{Z} \mid \langle \epsilon_k, \xi \rangle_{\mathbb{Z}} = 1\}.$$

By the construction, $\mathbf{k}(\xi)$ is a Maya diagram of charge r, and the map $\xi \mapsto \mathbf{k}(\xi)$ defines a bijection form $W_{\mathbb{Z}}\Lambda_r$ to $\mathcal{M}_{\mathbb{Z}}^{(r)}$. Hence we have a bijection $\Xi_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{M}_{\mathbb{Z}}$.

Note that $\Lambda_r \in \Xi_{\mathbb{Z}}$ and $\Lambda_r^c \in \Gamma_{\mathbb{Z}}$ are identified with the (complementary) ground states:

$$\mathbf{k}(\Lambda_r) = \mathbb{Z}_{\le r} \quad \text{and} \quad \mathbf{k}(\Lambda_r^c) = \mathbb{Z}_{>r}.$$

Let us recall the action of σ_i, $i \in \mathbb{Z}$, on $\Xi_{\mathbb{Z}}$ (*resp.*, $\Gamma_{\mathbb{Z}}$). Under the identification $\Xi_{\mathbb{Z}} \cong \mathcal{M}_{\mathbb{Z}}$ (*resp.*, $\Gamma_{\mathbb{Z}} \cong \mathcal{M}_{\mathbb{Z}}^c$), there is the induced action of σ_i on $\mathcal{M}_{\mathbb{Z}}$ (*resp.*, $\mathcal{M}_{\mathbb{Z}}^c$). It is easy to see that the explicit form of the action is just the transposition $(i, i+1)$ on $\mathbb{Z}$.

3.3. **Definition of BZ data associated to $\mathbb{Z}$.** Let $I = [n+1, n+m]$ be a finite interval in $\mathbb{Z}$. Set

$$\mathcal{M}_{\mathbb{Z}}(I) := \left\{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}} \mid \mathbf{k} = \mathbb{Z}_{\le n} \cup \mathbf{k}_I, \text{ for some } \mathbf{k}_I \in \mathcal{M}_I^{\times}\right\},$$
$$\mathcal{M}_{\mathbb{Z}}^c(I) := \left\{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c \mid \mathbf{k} = \mathbf{k}_I \cup \mathbb{Z}_{\ge n+m+2}, \text{ for some } \mathbf{k}_I \in \mathcal{M}_I^{\times}\right\}.$$

Define a map $\mathrm{res}_I : \mathcal{M}_{\mathbb{Z}}(I) \to \mathcal{M}_I^{\times}$ by $\mathbf{k} \mapsto \mathbf{k}_I$. It is obvious that res_I is a bijection. If we set $\Omega_I(\mathbf{k}) := (\mathrm{res}_I)^{-1}\big(\widetilde{I} \setminus \mathrm{res}_I(\mathbf{k})\big)$ for $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$, then $\Omega_I(\mathbf{k}) \in \mathcal{M}_{\mathbb{Z}}(I)$ and the map $\Omega_I : \mathcal{M}_{\mathbb{Z}}(I) \to \mathcal{M}_{\mathbb{Z}}(I)$ is also a bijection. Similarly, we define bijections $\mathrm{res}_I^c : \mathcal{M}_{\mathbb{Z}}^c(I) \xrightarrow{\sim} \mathcal{M}_I^{\times}$ and $\Omega_I^c : \mathcal{M}_{\mathbb{Z}}^c(I) \xrightarrow{\sim} \mathcal{M}_{\mathbb{Z}}^c(I)$. The following lemma is easily verified.

Lemma 3.3.1. (1) *A Maya diagram* $\mathbf{k}$ *belongs to* $\mathcal{M}_{\mathbb{Z}}(I)$ *if and only if* $\mathbf{k}^c$ *belongs to* $\mathcal{M}_{\mathbb{Z}}^c(I)$.
(2) *For each* $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$, *we have*

$$(\mathrm{res}_I^c)^{-1}\big(\widetilde{I} \setminus \mathrm{res}_I(\mathbf{k})\big) = \mathbf{k}^c, \qquad \mathrm{res}_I^{-1}\big(\widetilde{I} \setminus \mathrm{res}_I^c(\mathbf{k}^c)\big) = \mathbf{k}, \tag{3.3.1}$$

$$(\mathrm{res}_I^c)^{-1}\big(\mathrm{res}_I(\mathbf{k})\big) = \Omega_I^c(\mathbf{k}^c), \qquad \mathrm{res}_I^{-1}\big(\mathrm{res}_I^c(\mathbf{k}^c)\big) = \Omega_I(\mathbf{k}), \tag{3.3.2}$$

$$\big(\Omega_I(\mathbf{k})\big)^c = \Omega_I^c(\mathbf{k}^c), \qquad \big(\Omega_I^c(\mathbf{k}^c)\big)^c = \Omega_I(\mathbf{k}). \tag{3.3.3}$$

Let $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}}$ be a collection of integers indexed by $\mathcal{M}_{\mathbb{Z}}$. For such an $\mathbf{M}$, we define $\mathbf{M}_I := (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)}$. By the bijection $\mathrm{res}_I : \mathcal{M}_{\mathbb{Z}}(I) \xrightarrow{\sim} \mathcal{M}_I^{\times}$, $\mathbf{M}_I$ can be regarded as a collection of integers indexed by $\mathcal{M}_I^{\times}$. Similarly, for $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c}$, we define $\mathbf{M}_I := (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c(I)}$, which is regarded as a collection of integers indexed by $\mathcal{M}_I^{\times}$.

Definition 3.3.2. (1) *A collection* $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}^c_{\mathbb{Z}}}$ *of integers is called a complementary BZ (c-BZ for short) datum associated to* $\mathbb{Z}$ *if it satisfies the following conditions:*

(1-a) *For each finite interval* K *in* $\mathbb{Z}$, $\mathbf{M}_K = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}^\times_K}$ *is an element of* $\mathcal{BZ}_K$.

(1-b) *For each* $\mathbf{k} \in \mathcal{M}^c_{\mathbb{Z}}$, *there exist a finite interval* I *in* $\mathbb{Z}$ *such that*

(1-i) $\mathbf{k} \in \mathcal{M}^c_{\mathbb{Z}}(I)$,

(1-ii) *for every finite interval* $J \supset I$, $M_{\Omega^c_J(\mathbf{k})} = M_{\Omega^c_I(\mathbf{k})}$.

(2) *A collection* $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ *of integers is called an e-BZ datum associated to* $\mathbb{Z}$ *if it satisfies the following conditions:*

(2-a) *For each finite interval* K *in* $\mathbb{Z}$, $\mathbf{M}_K = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}^\times_K}$ *is an element of* $\mathcal{BZ}^e_K$.

(2-b) *For each* $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$, *there exist a finite interval* I *in* $\mathbb{Z}$ *such that*

(2-i) $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$,

(1-ii) *for every finite interval* $J \supset I$, $M_{\Omega_J(\mathbf{k})} = M_{\Omega_I(\mathbf{k})}$.

We denote by $\mathcal{BZ}_{\mathbb{Z}}$ (*resp.*, $\mathcal{BZ}^e_{\mathbb{Z}}$) the set of all c-BZ (*resp.*, e-BZ) data associated to $\mathbb{Z}$.

Let us fix a c-BZ datum $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}^c_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}$. For each complementary Maya diagram $\mathbf{k} \in \mathcal{M}^c_{\mathbb{Z}}$, we denote by $\mathrm{Int}^c(\mathbf{M};\mathbf{k})$ the set of all finite intervals I in $\mathbb{Z}$ satisfying the condition (1-b) in the definition above. For a given e-BZ datum $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}} \in \mathcal{BZ}^e_{\mathbb{Z}}$, we define $\mathrm{Int}^e(\mathbf{M};\mathbf{k})$ similarly. Note that, in the latter case, $\mathbf{k}$ is an element of $\mathcal{M}_{\mathbb{Z}}$.

Remark . Recall the identification $\mathcal{M}^c_{\mathbb{Z}} \cong \Gamma_{\mathbb{Z}}$. As mentioned before, a complementary Maya diagram $\mathbf{k} \in \mathcal{M}^c_{\mathbb{Z}}$ can be written as $\mathbf{k} = w\Lambda^c_r$ for $w \in W_{\mathbb{Z}}$ and $r \in \mathbb{Z}$. Then, the condition (1-b) in Definition 3.3.2 is rewritten as follows:

(1-b)' for each $w \in W_{\mathbb{Z}}$ and $r \in \mathbb{Z}$, there exist a finite interval I in $\mathbb{Z}$ such that

(1-i)' $r \in I$ and $w \in W_I$,

(1-ii)' for every finite interval $J \supset I$, $M_{w\Lambda^J_r} = M_{w\Lambda^I_r}$.

Namely, the $\mathcal{BZ}_{\mathbb{Z}}$ above coincides with the set of all BZ data of type A_∞ in the sense of [NSS].

For a given $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}^c_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}}$, introduce a new collection $\mathbf{M}^* = (M^*_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ of integers by

$$M^*_{\mathbf{k}} := M_{\mathbf{k}^c}.$$

As in the finite case, the following lemma is easily verified.

Lemma 3.3.3. *For* $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$, $\mathbf{M}^*$ *is an element of* $\mathcal{BZ}^e_{\mathbb{Z}}$. *Moreover, the map* $* : \mathcal{BZ}_{\mathbb{Z}} \to \mathcal{BZ}^e_{\mathbb{Z}}$ *is a bijection.*

The inverse of this bijection is also denoted by $*$.

For $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$, we introduce another collection $\Theta(\mathbf{M}) = (\Theta(M)_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ of integers as follows. Fix $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$ and take the complement $\mathbf{k}^c \in \mathcal{M}^c_{\mathbb{Z}}$ of $\mathbf{k}$. Since $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}$, there exists a finite interval $I \in \mathrm{Int}^c(\mathbf{M};\mathbf{k}^c)$. Note that, for such an I, we have $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$ by Lemma 3.3.1 (1). We define $\Theta(M)_{\mathbf{k}} := M_{(\mathrm{res}^c_I)^{-1}(\mathrm{res}_I(\mathbf{k}))}$. By (3.3.2) in Lemma 3.3.1, we have $(\mathrm{res}^c_I)^{-1}\big(\mathrm{res}_I(\mathbf{k})\big) = \Omega^c_I(\mathbf{k}^c)$. Therefore, by the condition (1-b) of Definition 3.3.2, the definition of $\Theta(\mathbf{M})$ does not depend on the choice of I.

Remark . Under the identification $\mathcal{M}^c_{\mathbb{Z}} \cong \Gamma_{\mathbb{Z}}$, the bijection above agrees with the map Θ, which is introduced in [NSS].

3.4. **Kashiwara operators on BZ data associated to $\mathbb{Z}$.** In [NSS], we defined the action of Kashiwara operators on $\mathcal{BZ}_{\mathbb{Z}}$. In this subsection, we reformulate some results of [NSS] under the identification $\mathcal{M}_{\mathbb{Z}}^{c} \cong \Gamma_{\mathbb{Z}}$.

First, we define the action of the raising Kashiwara operators $\widetilde{e}_p$, $p \in \mathbb{Z}$, on $\mathcal{BZ}_{\mathbb{Z}}$. For $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}} \in \mathcal{BZ}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, define

$$\varepsilon_p(\mathbf{M}) := -\left(\Theta(M)_{\mathbf{k}(\Lambda_p)} + \Theta(M)_{\mathbf{k}(\sigma_p\Lambda_p)} - \Theta(M)_{\mathbf{k}(\Lambda_{p+1})} - \Theta(M)_{\mathbf{k}(\Lambda_{p-1})}\right).$$

By the definition of $\mathcal{BZ}_{\mathbb{Z}}$, $\varepsilon_p(\mathbf{M})$ is a nonnegative integer. Indeed, take a finite interval I from $\mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\sigma_p\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_{p+1})^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_{p-1})^c)$. Then, by an argument similar to the one in [NSS], we get $\varepsilon_p(\mathbf{M}) = \varepsilon_p(\mathbf{M}_I)$. Hence this is a nonnegative integer.

If $\varepsilon_p(\mathbf{M}) = 0$, we set $\widetilde{e}_p\mathbf{M} = 0$. Suppose that $\varepsilon_p(\mathbf{M}) > 0$. Then we define $\widetilde{e}_p\mathbf{M} = (M'_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}}$ as follows. For $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}$, take a finite interval I in $\mathbb{Z}$ such that $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}(I)$ and $I \in \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\sigma_p\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_{p+1})^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_{p-1})^c)$. Set

$$M'_{\mathbf{k}} := \left(\widetilde{e}_p\mathbf{M}_I\right)_{\mathrm{res}_I^c(\mathbf{k})}.$$

Here we note that $\widetilde{e}_p\mathbf{M}_I$ is defined since $\mathbf{M}_I \in \mathcal{BZ}_I$.

Second, let us define the action of the lowering Kashiwara operators $\widetilde{f}_p$, $p \in \mathbb{Z}$, on $\mathcal{BZ}_{\mathbb{Z}}$. For $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}} \in \mathcal{BZ}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, we define $\widetilde{f}_p\mathbf{M} = (M''_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}}$ as follows. For $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}$, take a finite interval I in $\mathbb{Z}$ such that $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}(I)$ and $I \in \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\Lambda_p)^c) \cap \mathrm{Int}^c(\mathbf{M}; \mathbf{k}(\sigma_p\Lambda_p)^c)$. Set

$$M''_{\mathbf{k}} := \left(\widetilde{f}_p\mathbf{M}_I\right)_{\mathrm{res}_I^c(\mathbf{k})}.$$

Proposition 3.4.1 ([NSS])**.** (1) *The definition above of $M'_{\mathbf{k}}$ (resp., $M''_{\mathbf{k}}$) does not depend on the choice of I.*
(2) *For each $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}} \in \mathcal{BZ}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, $\widetilde{e}_p\mathbf{M}$ (resp., $\widetilde{f}_p\mathbf{M}$) is contained in $\mathcal{BZ}_{\mathbb{Z}} \cup \{0\}$ (resp., $\mathcal{BZ}_{\mathbb{Z}}$).*

For $\mathbf{M} \in \mathcal{B}Z_{\mathbb{Z}}^{e}$, set $\varepsilon_p^*(\mathbf{M}) := \varepsilon_p(\mathbf{M}^*)$, $p \in \mathbb{Z}$. Define the Kashiwara operators $\widetilde{e}_p^*$ and $\widetilde{f}_p^*$ on $\mathcal{B}Z_{\mathbb{Z}}^{e}$ by

$$\widetilde{e}_p^*\mathbf{M} := \begin{cases} \left(\widetilde{e}_p(\mathbf{M}^*)\right)^* & \text{if } \varepsilon_p^*(\mathbf{M}) > 0, \\ 0 & \text{if } \varepsilon_p^*(\mathbf{M}) = 0, \end{cases} \qquad \widetilde{f}_p^*\mathbf{M} := \left(\widetilde{f}_p(\mathbf{M}^*)\right)^*.$$

The following corollary is easily obtained Proposition 3.4.1.

Corollary 3.4.2. *For each $\mathbf{M} \in \mathcal{BZ}_{\mathbb{Z}}^{e}$ and $p \in \mathbb{Z}$, $\widetilde{e}_p^*\mathbf{M}$ (resp., $\widetilde{f}_p^*\mathbf{M}$) is contained in $\mathcal{BZ}_{\mathbb{Z}}^{e} \cup \{0\}$ (resp., $\mathcal{BZ}_{\mathbb{Z}}^{e}$).*

4. Berenstein-Zelevinsky data of type $A_{l-1}^{(1)}$

4.1. **Root datum of type $A_{l-1}^{(1)}$.** Let us recall the notation of [NSS]. Fix $l \in \mathbb{Z}_{\geq 3}$. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra of type $A_{l-1}^{(1)}$, $\widehat{\mathfrak{h}}$ the Cartan subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{h}_i \in \widehat{\mathfrak{h}}$, $i \in \widehat{I} := \{0, 1, \cdots, l-1\}$, the simple coroots of $\widehat{\mathfrak{g}}$, and $\widehat{\alpha}_i \in \widehat{\mathfrak{h}}^* := \mathrm{Hom}_{\mathbb{C}}(\widehat{\mathfrak{h}}, \mathbb{C})$, $i \in \widehat{I}$, the simple roots of $\widehat{\mathfrak{g}}$. Note that $\langle \widehat{h}_i, \widehat{\alpha}_j \rangle = \widehat{a}_{ij}$ for $i, j \in \widehat{I}$. Here, $\langle \cdot, \cdot \rangle : \widehat{\mathfrak{h}} \times \widehat{\mathfrak{h}}^* \to \mathbb{C}$ is the canonical pairing, and $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ is the Cartan matrix of type $A_l^{(1)}$ with index set $\widehat{I}$; the entries

$\widehat{a}_{ij}$ are given by

$$\widehat{a}_{ij} := \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1 \text{ or } l-1, \\ 0 & \text{otherwise.} \end{cases}$$

4.2. **Action of the affine Weyl group on Maya diagrams.** Consider a bijection $\tau : \mathbb{Z} \to \mathbb{Z}$ given by $\tau(j) := j+1$ for $j \in \mathbb{Z}$. It induces a $\mathbb{C}$-linear automorphism $\tau : \mathfrak{t}^* \overset{\sim}{\to} \mathfrak{t}^*$ such that $\tau(\Lambda_j) = \Lambda_{j+1}$ and $\tau(\Lambda_j^c) = \Lambda_{j+1}^c$ for all $j \in \mathbb{Z}$. Since $\alpha_j = -\Lambda_{j-1} + 2\Lambda_j - \Lambda_{j+1}$, we also have $\tau(\alpha_j) = \alpha_{j+1}$. Moreover, we have $\tau \circ \sigma_j = \sigma_{j+1} \circ \tau$. Hence we have an induced group automorphism $\tau : W_{\mathbb{Z}} \overset{\sim}{\to} W_{\mathbb{Z}}$ given by $\sigma_j \mapsto \tau \circ \sigma_j \circ \tau^{-1} = \sigma_{j+1}$. Also, the restriction of $\tau : \mathfrak{t}^* \overset{\sim}{\to} \mathfrak{t}^*$ to the subset $\Xi_{\mathbb{Z}}$ (*resp.*, $\Gamma_{\mathbb{Z}}$) gives rise to a bijection $\tau : \Xi_{\mathbb{Z}} \overset{\sim}{\to} \Xi_{\mathbb{Z}}$ (*resp.*, $\tau : \Gamma_{\mathbb{Z}} \overset{\sim}{\to} \Gamma_{\mathbb{Z}}$).

For $i \in \widehat{I}$, define a family S_i of automorphism of $\mathfrak{t}^*$ by

$$S_i := \{\sigma_{i+al} \mid a \in \mathbb{Z}\}.$$

Since $l \geq 3$, $\sigma_{j_1}\sigma_{j_2} = \sigma_{j_2}\sigma_{j_1}$ for all $\sigma_{j_1}, \sigma_{j_2} \in S_i$, and for a fixed $\gamma \in \Xi_{\mathbb{Z}}$ or $\Gamma_{\mathbb{Z}}$, there exists a finite subset $S_i(\gamma) \subset S_i$ such that $\sigma_j(\gamma) = \gamma$ for all $\sigma_j \in S_i \setminus S_i(\gamma)$. Therefore, we can define the following infinite product $\widehat{\sigma}_i$ of operators acting on $\Xi_{\mathbb{Z}}$ and on $\Gamma_{\mathbb{Z}}$:

$$\widehat{\sigma}_i := \prod_{\sigma_j \in S_i} \sigma_j.$$

We easily obtain the following lemma:

Lemma 4.2.1. (1) *For each $i \in \widehat{I}$, we have $\tau \circ \widehat{\sigma}_i = \widehat{\sigma}_{i+1} \circ \tau$. Here we regard $i \in \widehat{I}$ as an element of $\mathbb{Z}/l\mathbb{Z}$.*

(2) *Let $\widetilde{W}_{l-1}^{(1)}$ be the group generated by $\widehat{\sigma}_i$, $i \in \widehat{I}$, and τ. Then it is naturally isomorphic to the extended affine Weyl group of type $A_{l-1}^{(1)}$. Moreover, the subgroup $W_{l-1}^{(1)}$ of $\widetilde{W}_{l-1}^{(1)}$ generated by $\widehat{\sigma}_i$, $i \in \widehat{I}$, is naturally isomorphic to the affine Weyl group of type $A_{l-1}^{(1)}$.*

Let us recall the identifications $\Xi_{\mathbb{Z}} \cong \mathcal{M}_{\mathbb{Z}}$ and $\Gamma_{\mathbb{Z}} \cong \mathcal{M}_{\mathbb{Z}}^c$. Then the induced bijections $\tau : \mathcal{M}_{\mathbb{Z}}^\flat \overset{\sim}{\to} \mathcal{M}_{\mathbb{Z}}^\flat$ ($\flat = \phi$ or c) are given as:

$$\mathcal{M}_{\mathbb{Z}}^{(r)} \ni \mathbf{k} = \{k_j \mid j \in \mathbb{Z}_{\leq r}\} \mapsto \tau(\mathbf{k}) = \{k_j + 1 \mid j \in \mathbb{Z}_{\leq r}\} \in \mathcal{M}_{\mathbb{Z}}^{(r+1)},$$
$$\mathcal{M}_{\mathbb{Z}}^{(r),c} \ni \mathbf{k}' = \{k'_j \mid j \in \mathbb{Z}_{>r}\} \mapsto \tau(\mathbf{k}') = \{k'_j + 1 \mid j \in \mathbb{Z}_{>r}\} \in \mathcal{M}_{\mathbb{Z}}^{(r+1),c}.$$

Now, we give a quick review of the $\widetilde{W}_{l-1}^{(1)}$-action on $\mathcal{M}_{\mathbb{Z}}$. For details, see [N] for example. For $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$, consider an l-tuple of Maya diagrams $(\mathbf{k}^1, \mathbf{k}^2, \ldots, \mathbf{k}^l)$ by

$$\mathbf{k}^j := \{k \in \mathbb{Z} \mid (k-1)l + j \in \mathbf{k}\}, \quad 1 \leq j \leq l.$$

It is clear that the correspondence $\mathcal{M}_{\mathbb{Z}} \ni \mathbf{k} \mapsto (\mathbf{k}^1, \mathbf{k}^2, \ldots, \mathbf{k}^l) \in \left(\mathcal{M}_{\mathbb{Z}}\right)^l$ is bijective. Moreover, if $\mathbf{k}$ is of charge r and $\mathbf{k}^j$ is of charge r_j, then $r = r_1 + \cdots + r_l$. The action of $\widetilde{W}_{l-1}^{(1)}$ on $\mathcal{M}_{\mathbb{Z}}$ can be translated as follows:

Lemma 4.2.2. *We have*

$$\begin{aligned} \widehat{\sigma}_0(\mathbf{k}) &= (\tau(\mathbf{k}^l), \mathbf{k}^2, \ldots, \mathbf{k}^{l-1}, \tau^{-1}(\mathbf{k}^1)), \\ \widehat{\sigma}_i(\mathbf{k}) &= (\mathbf{k}^1, \ldots, \mathbf{k}^{i+1}, \mathbf{k}^i, \ldots, \mathbf{k}^l), \quad 1 \leq i \leq l-1, \\ \pi(\mathbf{k}) &= (\tau(\mathbf{k}^l), \mathbf{k}^1, \ldots, \mathbf{k}^{l-1}). \end{aligned}$$

Definition 4.2.3. *A Maya diagram $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{(r)}$ is called an l-core of charge r if $\mathbf{k}^j$ is the ground state of charge r_j for all $1 \leq j \leq l$.*

Proposition 4.2.4. *A Maya diagram $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{(r)}$ of charge r belongs to the $W_{l-1}^{(1)}$-orbit through the ground state $\mathbb{Z}_{\le r}$ if and only if $\mathbf{k}$ is an l-core of charge r.*

Let $Y = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n) \in \mathcal{Y}$ be a Young diagram. Then Y is realized as a collection of boxes arranged in left-justified rows, with λ_i-boxes in the i-th row. Each box in Y determines a hook, which consists of the box itself and all those boxes in its row to the right of the box or in its column below the box. The hook length of a box is the number of boxes in its hook. Then the following fact is well-known.

Proposition 4.2.5. *A Maya diagram $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{(r)}$ is called an l-core of charge r in the sense of Definition 4.2.3 if and only if the corresponding Young diagram $Y(\mathbf{k})$ contains no hook whose length is a multiple of l.*

Therefore, we obtain the following corollary.

Corollary 4.2.6. *A Maya diagram $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{(r)}$ of charge r belongs to the $W_{l-1}^{(1)}$-orbit through the ground state $\mathbb{Z}_{\le r}$ if and only if the corresponding Young diagram $Y(\mathbf{k})$ contains no hook whose length is a multiple of l.*

Remark . (1) Since an element of $W_{l-1}^{(1)}$ is an infinite product of elements of $W_{\mathbb{Z}}$, $W_{l-1}^{(1)}$ is not a subgroup of $W_{\mathbb{Z}}$.

(2) The set $\mathcal{M}_{\mathbb{Z}}^{(r)} = W_{\mathbb{Z}}\big(\mathbf{k}(\Lambda_r)\big)$ has infinitely many $W_{l-1}^{(1)}$-orbits.

4.3. **BZ data of type $A_{l-1}^{(1)}$.** We set $\sigma := \tau^l$. For $\mathbf{M} \in \mathcal{B}Z_{\mathbb{Z}}$, define new collections $\sigma(\mathbf{M})$ and $\sigma^{-1}(\mathbf{M})$ of integers indexed by $\mathcal{M}_{\mathbb{Z}}^c$ by $\sigma(\mathbf{M})_{\mathbf{k}} := \mathbf{M}_{\sigma^{-1}(\mathbf{k})}$ and $\sigma^{-1}(\mathbf{M})_{\mathbf{k}} := \mathbf{M}_{\sigma(\mathbf{k})}$ for each $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c$, respectively. It is shown in [NSS] that both $\sigma(\mathbf{M})$ and $\sigma^{-1}(\mathbf{M})$ are elements of $\mathcal{B}Z_{\mathbb{Z}}$.

Similarly, for $\mathbf{M} \in \mathcal{B}Z_{\mathbb{Z}}^e$, we can define new collections $\sigma^{\pm}(\mathbf{M})$, and prove that they are elements of $\mathcal{B}Z_{\mathbb{Z}}^e$.

Lemma 4.3.1 ([NSS])**.** (1) *On $\mathcal{B}Z_{\mathbb{Z}}$, we have $\Theta \circ \sigma = \sigma \circ \Theta$.*

(2) *For $\mathbf{M} \in \mathcal{B}Z_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, $\varepsilon_p(\sigma(\mathbf{M})) = \varepsilon_{\sigma^{-1}(p)}(\mathbf{M})$.*

(3) *There hold $\sigma \circ \widetilde{e}_p = \widetilde{e}_{\sigma(p)} \circ \sigma$ and $\sigma \circ \widetilde{f}_p = \widetilde{f}_{\sigma(p)} \circ \sigma$ on $\mathcal{B}Z_{\mathbb{Z}} \cup \{0\}$ for all $p \in \mathbb{Z}$. Here it is understood that $\sigma(0) = 0$.*

Definition 4.3.2. *Set*

$$\mathcal{B}Z_{\mathbb{Z}}^{\sigma} := \{\mathbf{M} \in \mathcal{B}Z_{\mathbb{Z}} \mid \sigma(\mathbf{M}) = \mathbf{M}\} \quad \text{and} \quad (\mathcal{B}Z_{\mathbb{Z}}^e)^{\sigma} := \{\mathbf{M} \in \mathcal{B}Z_{\mathbb{Z}}^c \mid \sigma(\mathbf{M}) = \mathbf{M}\}.$$

An element $\mathbf{M}$ of $\mathcal{B}Z_{\mathbb{Z}}^{\sigma}$ (resp., $(\mathcal{B}Z_{\mathbb{Z}}^e)^{\sigma}$) is called a c-BZ (resp., e-BZ)datum of type $A_{l-1}^{(1)}$.

Now we define a crystal structure on $\mathcal{B}Z_{\mathbb{Z}}^{\sigma}$, following [NSS]. For $\mathbf{M} \in \mathcal{B}Z_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$, we set

$$\mathrm{wt}(\mathbf{M}) := \sum_{p \in \widehat{I}} \Theta(\mathbf{M})_{\mathbf{k}(\Lambda_p)} \widehat{\alpha}_p, \quad \widehat{\varepsilon}_p(\mathbf{M}) := \varepsilon_p(\mathbf{M}), \quad \widehat{\varphi}_p(\mathbf{M}) := \widehat{\varepsilon}_p(\mathbf{M}) + \langle \widehat{h}_p, \mathrm{wt}(\mathbf{M}) \rangle.$$

In order to define the action of Kashiwara operators, we need the following.

Lemma 4.3.3 ([NSS])**.** *Let $q, q' \in \mathbb{Z}$ with $|q - q'| \ge 2$. Then, we have $\widetilde{e}_q \widetilde{e}_{p'} = \widetilde{e}_{q'} \widetilde{e}_q$ and $\widetilde{f}_q \widetilde{f}_{q'} = \widetilde{f}_{q'} \widetilde{f}_q$, as operators from $\mathcal{B}Z_{\mathbb{Z}}$ to $\mathcal{B}Z_{\mathbb{Z}} \cup \{0\}$.*

For $\mathbf{M} \in \mathcal{B}Z_{\mathbb{Z}}^{\sigma}$ and $p \in \widehat{I}$, we define $\widehat{e}_p \mathbf{M}$ and $\widehat{f}_p \mathbf{M}$ as follows. If $\widehat{\varepsilon}_p(\mathbf{M}) = 0$, we set $\widehat{e}_p \mathbf{M} := 0$. If $\widehat{\varepsilon}_p(\mathbf{M}) > 0$, then we define a new collection $\widehat{e}_p \mathbf{M} = (M'_{\mathbf{k}})$ of integers indexed by $\mathcal{M}_{\mathbb{Z}}^c$ by

$$M'_{\mathbf{k}} := \big(e_{L(\mathbf{k},p)} \mathbf{M}\big)_{\mathbf{k}} \quad \text{for each } \mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^c.$$

Here, $L(\mathbf{k},p) := \{q \in p + l\mathbb{Z} \mid q \in \mathbf{k} \text{ and } q+1 \notin \mathbf{k}\}$ and $e_{L(\mathbf{k},p)} := \prod_{q \in L(\mathbf{k},p)} \widetilde{e}_q$. By the definition, $L(\mathbf{k},p)$ is a finite set such that $|q-q'| > 2$ for all $q, q' \in L(\mathbf{k},p)$ with $q \neq q'$. Therefore, by Lemma 4.3.3, $e_{L(\mathbf{k},p)}$ is a well-defined operator on $\mathcal{B}Z_{\mathbb{Z}}$.

A collection $\widehat{f}_p\mathbf{M} = (M''_{\mathbf{k}})$ of integers indexed by $\mathcal{M}^c_{\mathbb{Z}}$ is defined by

$$M''_{\mathbf{k}} := \left(f_{L(\mathbf{k},p)}\mathbf{M}\right)_{\mathbf{k}} \quad \text{for each } \mathbf{k} \in \mathcal{M}^c_{\mathbb{Z}},$$

where $f_{L(\mathbf{k},p)} := \prod_{q \in L(\mathbf{k},p)} \widetilde{f}_q$. By the same reasoning as above, we see that $f_{L(\mathbf{k},p)}$ is a well-defined operator on $\mathcal{B}Z_{\mathbb{Z}}$.

Proposition 4.3.4 ([NSS])**.** (1) *We have* $\widehat{e}_p\mathbf{M} \in \mathcal{B}Z^{\sigma}_{\mathbb{Z}} \cup \{0\}$ *and* $\widehat{f}_p\mathbf{M} \in \mathcal{B}Z^{\sigma}_{\mathbb{Z}}$*.*

(2) *The set* $\mathcal{B}Z^{\sigma}_{\mathbb{Z}}$*, equipped with the maps* $\mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p$*, is a crystal of type* $A^{(1)}_{l-1}$*.*

Let $\mathbf{O}$ be a collection of integers indexed by $\mathcal{M}^c_{\mathbb{Z}}$ whose $\mathbf{k}$-component is equal to 0 for all $\mathbf{k} \in \mathcal{M}^c_{\mathbb{Z}}$. It is obvious that $\mathbf{O} \in \mathcal{B}Z^{\sigma}_{\mathbb{Z}}$. Let $\mathcal{B}Z^{\sigma}_{\mathbb{Z}}(\mathbf{O})$ denote the connected component of the crystal $\mathcal{B}Z^{\sigma}_{\mathbb{Z}}$ containing $\mathbf{O}$. The following is the main result of [NSS].

Theorem 4.3.5 ([NSS])**.** *As a crystal,* $\left(\mathcal{B}Z^{\sigma}_{\mathbb{Z}}(\mathbf{O}); \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p\right)$ *is isomorphic to* $B(\infty)$ *of type* $A^{(1)}_{l-1}$*.*

In a way similar to the finite case, we can define a crystal structure on $(\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}$. By the construction, it is easy to see that $* \circ \sigma = \sigma \circ *$. Therefore, the restriction of $* : \mathcal{B}Z_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{B}Z^e_{\mathbb{Z}}$ to the subset $\mathcal{B}Z^{\sigma}_{\mathbb{Z}}$ gives rise to a bijection $* : \mathcal{B}Z^{\sigma}_{\mathbb{Z}} \xrightarrow{\sim} (\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}$. We denote by $\mathbf{O}^*$ the image of $\mathbf{O} \in \mathcal{B}Z^{\sigma}_{\mathbb{Z}}$ under the bijection $*$. Then, $\mathbf{O}^*$ is a collection of integers indexed by $\mathcal{M}_{\mathbb{Z}}$ whose $\mathbf{k}$-component is equal to 0 for all $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$.

For $\mathbf{M} = (\mathcal{B}\mathcal{Z}^e_{\mathbb{Z}})^{\sigma}$ and $p \in \mathbb{Z}$, we define

$$\mathrm{wt}(\mathbf{M}) := \mathrm{wt}(\mathbf{M}^*), \quad \widehat{\varepsilon}^*_p(\mathbf{M}) := \widehat{\varepsilon}_p(\mathbf{M}^*), \quad \widehat{\varphi}^*_p(\mathbf{M}) := \widehat{\varepsilon}^*_p(\mathbf{M}) + \langle \widehat{h}_p, \mathrm{wt}(\mathbf{M})\rangle$$

and

$$\widehat{e}^*_p\mathbf{M} := \begin{cases} (\widehat{e}_p(\mathbf{M}^*))^* & \text{if } \widehat{\varepsilon}^*_p(\mathbf{M}) > 0, \\ 0 & \text{if } \widehat{\varepsilon}^*_p(\mathbf{M}) = 0, \end{cases} \qquad \widehat{f}^*_p := (\widehat{f}_p(\mathbf{M}^*))^*.$$

The following is an easy consequence of Theorem 4.3.5.

Corollary 4.3.6. (1) *The set* $(\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}$*, equipped with the maps* $\mathrm{wt}, \widehat{\varepsilon}^*_p, \widehat{\varphi}^*_p, \widehat{e}^*_p, \widehat{f}^*_p$*, is a crystal of type* $A^{(1)}_{l-1}$*.*

(2) *Let* $(\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}(\mathbf{O}^*)$ *denote the connected component of the crystal* $(\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}$ *containing* $\mathbf{O}^* \in (\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}$*. Then,* $\left((\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}(\mathbf{O}^*); \mathrm{wt}, \widehat{\varepsilon}^*_p, \widehat{\varphi}^*_p, \widehat{e}^*_p, \widehat{f}^*_p\right)$ *is isomorphic to* $B(\infty)$ *of type* $A^{(1)}_{l-1}$*.*

5. Lusztig data of infinite size and a crystal structure on them

5.1. Definition of Lusztig data.

Definition 5.1.1. (1) *Let* $\Delta^+_{\mathbb{Z}} := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i < j\}$*. A collection* $\mathbf{a} = (a_{i,j})_{(i,j) \in \Delta^+_{\mathbb{Z}}}$ *of nonnegative integers indexed by* $\Delta^+_{\mathbb{Z}}$ *is called a Lusztig datum associated to* $\mathbb{Z}$ *if there exist* $N = N_{\mathbf{a}} > 0$ *such that*

$$a_{i,j} = 0 \quad \text{for} \quad j - i \geq N_{\mathbf{a}}. \tag{5.1.1}$$

We denote by $\mathcal{B}_{\mathbb{Z}}$ *the set of all Lusztig data associated to* $\mathbb{Z}$*.*

(2) *For* $l \in \mathbb{Z}_{\geq 3}$*, a Lusztig datum* $\mathbf{a} = (a_{i,j})_{(i,j) \in \Delta^+_{\mathbb{Z}}} \in \mathcal{B}_{\mathbb{Z}}$ *is said to be of type* $A^{(1)}_{l-1}$ *if*

$$a_{i,j} = a_{i+l,j+l} \text{ for all } (i,j) \in \Delta^+_{\mathbb{Z}}. \tag{5.1.2}$$

We denote by $\mathcal{B}_{l-1}^{(1)}$ *the set of all Lusztig data of type* $A_{l-1}^{(1)}$.

(3) *A Lusztig datum* $\mathbf{a} \in \mathcal{B}_{l-1}^{(1)}$ *is said to be aperiodic if the following conditions are satisfied: for each* $(i,j) \in \Delta_{\mathbb{Z}}^{+}$, *there exists at least one* 0 *in the* l*-tuple of nonnegative integers*

$$\{a_{i,j}, a_{i+1,j+1}, \ldots, a_{i+l-1,j+l-1}\}.$$

We denote by $\mathcal{B}_{l-1}^{(1),ap}$ *the set of all aperiodic Lusztig data.*

The set $\mathcal{B}_{l-1}^{(1)}$ can be identified with the set of *multisegments*.

Definition 5.1.2. (1) *A segment of length* r *over* $\mathbb{Z}/l\mathbb{Z}$ *is a sequence of* r *consecutive elements in* $\mathbb{Z}/l\mathbb{Z}$*:*

$$\boxed{x_1}\boxed{x_2}\boxed{\quad\cdots\quad}\boxed{x_r}\ ,$$

where $x_p = i + p - 1,\ 1 \le p \le r$, *for some* $i \in \mathbb{Z}/l\mathbb{Z}$.

(2) *A multisegment over* $\mathbb{Z}/l\mathbb{Z}$ *is a multiset of segments over* $\mathbb{Z}/l\mathbb{Z}$. *The set of all multisegments over* $\mathbb{Z}/l\mathbb{Z}$ *is denoted by* $\mathbf{Seg}(\mathbb{Z}/l\mathbb{Z})$.

To $(i,j) \in \Delta_{\mathbb{Z}}^{+}$, we associate the segment of length $r = j - i$ with $x_1 = i \bmod l\mathbb{Z}$. Thus we can construct a bijection form $\mathcal{B}_{l-1}^{(1)}$ to $\mathbf{Seg}(\mathbb{Z}/l\mathbb{Z})$. Note that, for each $(i,j) \in \Delta_{\mathbb{Z}}^{+}$, $a_{i,j}$ is just the multiplicity of the corresponding segment.

Under the above identification $\mathcal{B}_{l-1}^{(1)} \cong \mathbf{Seg}(\mathbb{Z}/l\mathbb{Z})$, $\mathbf{a} \in \mathcal{B}_{l-1}^{(1)}$ is aperiodic if and only if the corresponding multisegment is aperiodic in the sense of Lusztig ([L3]).

5.2. Kashiwara operators on $\mathcal{B}_{\mathbb{Z}}$.

For $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}$ and $p \in \mathbb{Z}$, we set

$$A_k^{(p)}(\mathbf{a}) := \sum_{s \le k} (a_{s,p+1} - a_{s-1,p}) \quad \text{for } k \le p,$$

$$A_k^{*(p)}(\mathbf{a}) := \sum_{t \ge k+1} (a_{p,t} - a_{p+1,t+1}) \quad \text{for } k \ge p.$$

Because of the finiteness condition (5.1.1), each of the sums above is a finite sum, and there exist k_1 and k_2 such that

$$A_k^{(p)}(\mathbf{a}) = 0 \quad \text{for } k \le k_1, \qquad A_k^{*(p)}(\mathbf{a}) = 0 \quad \text{for } k \ge k_2.$$

If we set

$$\varepsilon_p(\mathbf{a}) := \max\left\{ A_k^{(p)}(\mathbf{a}) \;\middle|\; k \le p \right\}, \qquad \varepsilon_p^*(\mathbf{a}) := \max\left\{ A_k^{*(p)}(\mathbf{a}) \;\middle|\; k \ge p \right\},$$

then these are nonnegative integers. Also, set

$$\mathcal{K}(p;\mathbf{a}) := \left\{ k \;\middle|\; k \le p,\ \varepsilon_p(\mathbf{a}) = A_k^{(p)}(\mathbf{a}) \right\}, \quad \mathcal{K}^*(p;\mathbf{a}) := \left\{ k \;\middle|\; k \ge p,\ \varepsilon_p^*(\mathbf{a}) = A_k^{*(p)}(\mathbf{a}) \right\}.$$

Since $\mathcal{K}(p;\mathbf{a})$ is bounded above, we can define $k_f = k_f(p;\mathbf{a}) := \max\{k \mid k \in \mathcal{K}(p;\mathbf{a})\}$. Note that $\mathcal{K}(p;\mathbf{a})$ is not bounded below in general. However, if $\varepsilon_p(\mathbf{a}) > 0$, then it is a finite set. Therefore, we can define $k_e = k_e(p;\mathbf{a}) := \min\{k \mid k \in \mathcal{K}(p;\mathbf{a})\}$ only for $\mathbf{a}$ with $\varepsilon_p(\mathbf{a}) > 0$. Similarly, we define $k_e^* = k_e^*(p;\mathbf{a}) := \max\{k \mid k \in \mathcal{K}^*(p;\mathbf{a})\}$ only for $\mathbf{a}$ with $\varepsilon_p^*(\mathbf{a}) > 0$, and $k_f^* = k_f^*(p;\mathbf{a}) := \min\{k \mid k \in \mathcal{K}^*(p;\mathbf{a})\}$ for arbitrary $\mathbf{a}$.

Now we define Kashiwara operators $\widetilde{e}_p$, $\widetilde{f}_p$, $\widetilde{e}_p^*$, $\widetilde{f}_p^*$, $p \in \mathbb{Z}$, on $\mathcal{B}_{\mathbb{Z}}$ in a way similar to the finite case.

Let $I = [n+1, n+m]$ be a finite interval in $\mathbb{Z}$. Then, the set $\Delta_I^+ = \{(i,j) \mid i,j \in \widetilde{I} \text{ with } i < j\}$ is naturally regarded as a subset of $\Delta_{\mathbb{Z}}^+$. For a Lusztig datum $\mathbf{a} = (a_{i,j})_{(i,j)\in\Delta_{\mathbb{Z}}^+} \in \mathcal{B}_{\mathbb{Z}}$, we set $\mathbf{a}^I := (a_{i,j})_{(i,j)\in\Delta_I^+}$; it is an element of $\mathcal{B}_I$.

Let $p \in \mathbb{Z}$ and $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}$. Then, for each finite interval I with $I \supset [p - N_{\mathbf{a}}, p + N_{\mathbf{a}}]$, we have

$$\varepsilon_p(\mathbf{a}) = \varepsilon_p(\mathbf{a}^I) \quad \text{and} \quad \varepsilon_p^*(\mathbf{a}) = \varepsilon_p^*(\mathbf{a}^I). \tag{5.2.1}$$

Take a sufficiently large interval I, and assume that $\varepsilon_p(\mathbf{a}) = \varepsilon_p(\mathbf{a}^I) > 0$. Write $\widetilde{e}_p\mathbf{a}^I = (a'_{i,j})_{(i,j)\in\Delta_I^+} \in \mathcal{B}_I$, and define a new collection $\mathrm{Ind}_I^{\mathbb{Z}}(\widetilde{e}_p\mathbf{a}^I) = (a''_{i,j})_{(i,j)\in\Delta_{\mathbb{Z}}^+}$ of nonnegative integers by

$$a''_{i,j} := \begin{cases} a'_{i,j} & \text{if } (i,j) \in \Delta_I^+, \\ a_{i,j} & \text{otherwise.} \end{cases}$$

We also define $\mathrm{Ind}_I^{\mathbb{Z}}(\widetilde{f}_p\mathbf{a}^I)$, $\mathrm{Ind}_I^{\mathbb{Z}}(\widetilde{e}_p^*\mathbf{a}^I)$, and $\mathrm{Ind}_I^{\mathbb{Z}}(\widetilde{f}_p^*\mathbf{a}^I)$ in a similar way. It is obvious that

$$\widetilde{e}_p\mathbf{a} = \mathrm{Ind}_I^{\mathbb{Z}}(\widetilde{e}_p\mathbf{a}^I), \quad \widetilde{f}_p\mathbf{a} = \mathrm{Ind}_I^{\mathbb{Z}}(\widetilde{f}_p\mathbf{a}^I), \quad \widetilde{e}_p^*\mathbf{a} = \mathrm{Ind}_I^{\mathbb{Z}}(\widetilde{e}_p^*\mathbf{a}^I), \quad \widetilde{f}_p^*\mathbf{a} = \mathrm{Ind}_I^{\mathbb{Z}}(\widetilde{f}_p^*\mathbf{a}^I). \tag{5.2.2}$$

Remark . Let $\mathcal{B}_{\mathbb{Z}}^{fin}$ be the set of those Lusztig data $\mathbf{a} = (a_{i,j})_{(i,j)\in\Delta_{\mathbb{Z}}^+}$ associated to $\mathbb{Z}$ for which

$$a_{i,j} = 0 \quad \text{except for finitely many } (i,j) \in \Delta_{\mathbb{Z}}^+. \tag{5.2.3}$$

The (proper) subset $\mathcal{B}_{\mathbb{Z}}^{fin}$ of $\mathcal{B}_{\mathbb{Z}}$ should be called the set of Lusztig data of type A_∞. In the following, we will explain the reason.

A Lusztig datum $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}^{fin}$ can be regarded as an element of $\mathcal{B}_I$ for a sufficiently large interval I, and the set $\mathcal{B}_{\mathbb{Z}}^{fin} \cup \{0\}$ is stable under the Kashiwara operators $\widetilde{e}_p, \widetilde{f}_p, \widetilde{e}_p^*, \widetilde{f}_p^*,\ p \in \mathbb{Z}$. Furthermore, we can define a weight of $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}^{fin}$ by

$$\mathrm{wt}(\mathbf{a}) := -\sum_{p\in\mathbb{Z}} \left(\sum_{s\le p} \sum_{t\ge p+1} a_{s,t} \right) \alpha_p.$$

We remark that the right-hand side is a finite sum because of the finiteness condition (5.2.3). Hence each of $\left(\mathcal{B}_{\mathbb{Z}}^{fin}; \mathrm{wt}, \varepsilon_p, \varphi_p, \widetilde{e}_p, \widetilde{f}_p\right)$ and $\left(\mathcal{B}_{\mathbb{Z}}^{fin}; \mathrm{wt}, \varepsilon_p^*, \varphi_p^*, \widetilde{e}_p^*, \widetilde{f}_p^*\right)$ is a crystal of type A_∞. Moreover, they are both isomorphic to $B(\infty)$ of type A_∞. Indeed, it can be checked directly that they satisfy the conditions which uniquely characterize $B(\infty)$ ([KS]). However, since we do not use $\mathcal{B}_{\mathbb{Z}}^{fin}$ for the remainder of this paper, we omit the details.

Enomoto and Kashiwara [EK] gave a combinatorial description of $B(\infty)$ of type A_∞ by using the PBW basis. The crystal structure of $B(\infty)$ introduced by them agrees with our $\left(\mathcal{B}_{\mathbb{Z}}^{fin}; \mathrm{wt}, \varepsilon_p^*, \varphi_p^*, \widetilde{e}_p^*, \widetilde{f}_p^*\right)$.

5.3. **Crystal structure on $\mathcal{B}_{l-1}^{(1)}$.** Let us define a crystal structure of type $A_{l-1}^{(1)}$ on $\mathcal{B}_{l-1}^{(1)}$. The following lemma is easily shown.

Lemma 5.3.1. *Let $p \in \widehat{I} = \{0, 1, \cdots, l-1\}$ and $\mathbf{a} \in \mathcal{B}_{l-1}^{(1)}$. For each $r \in \mathbb{Z}$, we have*

$$A_k^{(p)} = A_{k+rl}^{(p+rl)}, \quad A_k^{*(p)} = A_{k+rl}^{*(p+rl)}, \quad \varepsilon_p(\mathbf{a}) = \varepsilon_{p+rl}(\mathbf{a}), \quad \varepsilon_p^*(\mathbf{a}) = \varepsilon_{p+rl}^*(\mathbf{a})$$

and

$$k_e(p+rl; \mathbf{a}) = k_e(p; \mathbf{a}) + rl, \qquad k_f(p+rl; \mathbf{a}) = k_f(p; \mathbf{a}) + rl,$$
$$k_e^*(p+rl; \mathbf{a}) = k_e^*(p; \mathbf{a}) + rl, \qquad k_f^*(p+rl; \mathbf{a}) = k_f^*(p; \mathbf{a}) + rl.$$

We now set

$$\mathrm{wt}(\mathbf{a}) := -\sum_{p\in\widehat{I}} r_p(\mathbf{a})\widehat{\alpha}_p, \qquad \text{where } r_p(\mathbf{a}) := \sum_{s\le p}\sum_{t\ge p+1} a_{s,t},$$

$$\widehat{\varepsilon}_p(\mathbf{a}) := \varepsilon_p(\mathbf{a}), \quad \widehat{\varepsilon}^*_p(\mathbf{a}) := \varepsilon^*_p(\mathbf{a}), \quad \widehat{\varphi}_p := \widehat{\varepsilon}_p(\mathbf{a}) + \langle \widehat{h}_p, \mathrm{wt}(\mathbf{a})\rangle, \quad \widehat{\varphi}^*_p := \widehat{\varepsilon}^*_p(\mathbf{a}) + \langle \widehat{h}_p, \mathrm{wt}(\mathbf{a})\rangle.$$

Note that, because of the finiteness condition (5.1.1), the right-hand side of the definition of $r_p(\mathbf{a})$ is a finite sum. If we take a sufficiently large interval I, then the following is obvious from the definitions:

$$\langle \widehat{h}_p, \mathrm{wt}(\mathbf{a})\rangle = \langle h_p, \mathrm{wt}(\mathbf{a}^I)\rangle. \tag{5.3.1}$$

Moreover, the next formulas follow immediately:

$$\widetilde{e}_q\widetilde{e}_{q'} = \widetilde{e}_{q'}\widetilde{e}_q, \quad \widetilde{f}_q\widetilde{f}_{q'} = \widetilde{f}_{q'}\widetilde{f}_q, \quad \widetilde{e}^*_q\widetilde{e}^*_{q'} = \widetilde{e}^*_{q'}\widetilde{e}^*_q, \quad \widetilde{f}^*_q\widetilde{f}^*_{q'} = \widetilde{f}^*_{q'}\widetilde{f}^*_q$$

for $q, q' \in \mathbb{Z}$ with $|q-q'| > 2$. Therefore, we can define (well-defined) operators by

$$\widehat{e}_p := \prod_{r\in\mathbb{Z}} \widetilde{e}_{p+rl}, \quad \widehat{f}_p := \prod_{r\in\mathbb{Z}} \widetilde{f}_{p+rl}, \quad \widehat{e}^*_p := \prod_{r\in\mathbb{Z}} \widetilde{e}^*_{p+rl}, \quad \widehat{f}^*_p := \prod_{r\in\mathbb{Z}} \widetilde{f}^*_{p+rl}.$$

By Lemma 5.3.1, the image of $\mathbf{a} \in \mathcal{B}^{(1)}_{l-1}$ under each of the operators above belongs to $\mathcal{B}^{(1)}_{l-1} \cup \{0\}$.

Proposition 5.3.2. *Each of* $\left(\mathcal{B}^{(1)}_{l-1}; \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p\right)$ *and* $\left(\mathcal{B}^{(1)}_{l-1}; \mathrm{wt}, \widehat{\varepsilon}^*_p, \widehat{\varphi}^*_p, \widehat{e}^*_p, \widehat{f}^*_p\right)$ *is a crystal of type* $A^{(1)}_{l-1}$.

Since one can easily check the axioms of a crystal (see [HK], for example) by using the definitions, we omit the details of the proof of this proposition.

Definition 5.3.3. *A Lusztig datum* $\mathbf{a} \in \mathcal{B}^{(1)}_{l-1}$ *is called a maximal element if* $\widehat{\varepsilon}_p(\mathbf{a}) = 0$ *for all* $p \in \widehat{I}$.

Let $\mathbf{z} = (z_1, z_2, \cdots)$ be an infinite series of nonnegative integers such that $z_n = 0$ for $n \gg 1$. Denote by $\mathcal{Z}$ the set of all such $\mathbf{z}$. For $\mathbf{z} \in \mathcal{Z}$, we define $\mathbf{a}_{\mathbf{z}} = ((a_{\mathbf{z}})_{i,j})_{(i,j)\in\Delta^+_{\mathbb{Z}}} \in \mathcal{B}^{(1)}_{l-1}$ by $(a_{\mathbf{z}})_{i,j} := z_{j-i}$. For $\mathbf{a} \in \mathcal{B}^{(1)}_{l-1}$, we introduce an infinite series $\mathbf{z}(\mathbf{a}) = (z(\mathbf{a})_1, z(\mathbf{a})_2, \cdots) \in \mathcal{Z}$ of nonnegative integers by $z(\mathbf{a})_n := \min\{a_{i,j} \mid j-i = n\}$ for $n \ge 1$, and set $\mathcal{B}^{(1)}_{l-1}(\mathbf{z}) := \{\mathbf{a} \in \mathcal{B}^{(1)}_{l-1} \mid \mathbf{z}(\mathbf{a}) = \mathbf{z}\}$. Then the following are obvious from the definitions:

$$\mathcal{B}^{(1)}_{l-1} = \bigsqcup_{\mathbf{z}\in\mathcal{Z}} \mathcal{B}^{(1)}_{l-1}(\mathbf{z}) \tag{5.3.2}$$

and

$$\mathcal{B}^{(1)}_{l-1}(\mathbf{0}) = \mathcal{B}^{(1),ap}_{l-1}, \quad \text{where we set } \mathbf{0} := (0, 0, \cdots) \in \mathcal{Z}.$$

Lemma 5.3.4. (1) *Denote by* $\mathrm{Max}(\mathcal{B}^{(1)}_{l-1})$ *the set of all maximal elements in* $\mathcal{B}^{(1)}_{l-1}$. *Then we have* $\mathrm{Max}(\mathcal{B}^{(1)}_{l-1}) = \{\mathbf{a}_{\mathbf{z}} \mid \mathbf{z} \in \mathcal{Z}\}$.

(2) *A Lusztig datum* $\mathbf{a} \in \mathcal{B}^{(1)}_{l-1}$ *is a maximal element if and only if* $\widehat{\varepsilon}^*_p(\mathbf{a}) = 0$ *for all* $p \in \widehat{I}$.

Proof. By the definitions, we have $\mathbf{a}_{\mathbf{z}} \in \mathrm{Max}(\mathcal{B}^{(1)}_{l-1})$. Conversely, let $\mathbf{a} \in \mathcal{B}^{(1)}_{l-1}$ be a maximal element. By the decomposition (5.3.2), there exists a unique $\mathbf{z} \in \mathcal{Z}$ such that $\mathbf{a} \in \mathcal{B}^{(1)}_{l-1}(\mathbf{z})$. What we need to prove is that $\mathbf{a} = \mathbf{a}_{\mathbf{z}}$. Introduce a new Lusztig datum $\mathbf{a}(0) = (a(0)_{i,j}) \in$

$\mathcal{B}^{(1)}_{l-1}(\mathbf{0})$ by $a(0)_{i,j} := a_{i,j} - z_{j-i}$ for each $(i,j) \in \Delta^+_{\mathbb{Z}}$. By the definitions, we have $\widehat{\varepsilon}_p(\mathbf{a}) = \widehat{\varepsilon}_p(\mathbf{a}(0))$ for all $p \in \widehat{I}$. Therefore, we may assume that $\mathbf{z} = \mathbf{0}$.

Let $\mathbf{a} \in \mathcal{B}^{(1)}_{l-1}(\mathbf{0}) = \mathcal{B}^{(1),ap}_{l-1}$. We will prove that *if* $\mathbf{a} \neq \mathbf{a_0}$, *then there exists* $p \in \widehat{I}$ *such that* $\widehat{\varepsilon}_p(\mathbf{a}) > 0$. Set $B_k(\mathbf{a}) = \max\{a_{k,1}, a_{k+1,2}, \ldots, a_{k+l-1,l}\}$ for $k \leq 0$. Since $B_k(\mathbf{a}) = 0$ for all sufficiently small k, there exists the minimum $k_0 \leq 0$ for which $B_{k_0}(\mathbf{a}) > 0$. Because of the aperiodicity condition for $\mathbf{a}$, there exists $p_0 \in \widehat{I}$ such that $a_{k_0+p_0-1,p_0} = 0$ and $a_{k_0+p_0,p_0+1} > 0$. Here the indices are regarded as elements of $\mathbb{Z}/l\mathbb{Z}$. Hence we have

$$A^{(p_0)}_{k_0+p_0}(\mathbf{a}) = \sum_{s \leq k_0+p_0} (a_{s,p+1} - a_{s-1,p}) = a_{k_0+p_0,p_0+1} > 0.$$

Therefore, we deduce that $\widehat{\varepsilon}_p(\mathbf{a}) = \varepsilon_{\mathbf{p}}(\mathbf{a}) > \mathbf{0}$. This proves part (1).

Let $\mathrm{Max}^*(\mathcal{B}^{(1)}_{l-1})$ be the set of those elements $\mathbf{a}$ in $\mathcal{B}^{(1)}_{l-1}$ such that $\widehat{\varepsilon}^*_p(\mathbf{a}) = 0$ for all $\widehat{I}$. By arguing as in the proof of part (1), we obtain $\mathrm{Max}^*(\mathcal{B}^{(1)}_{l-1}) = \{\mathbf{a_z} \mid \mathbf{z} \in \mathcal{Z}\}$. This proves part (2). □

The following corollary is an easy consequence of the lemma above.

Corollary 5.3.5. *For each* $\mathbf{z} = (z_n) \in \mathcal{Z}$, $\mathcal{B}^{(1)}_{l-1}(\mathbf{z})$ *contains a unique maximal element* $\mathbf{a_z}$ *for which* $\mathrm{wt}(\mathbf{a_z}) = -m(\mathbf{z})\delta$. *Here,* $m(\mathbf{z}) = \sum_{n\geq 1} nz_n$, *and* $\delta := \sum_{p\in\widehat{I}} \widehat{\alpha}_p$ *is the null root.*

Let $\widehat{P}$ denote the weight lattice of type $A^{(1)}_{l-1}$, and $\widehat{\lambda} \in \widehat{P}$. Consider the set $T_{\widehat{\lambda}} = \{t_{\widehat{\lambda}}\}$, and introduce two crystal structures $(T_{\widehat{\lambda}}; \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p)$ and $(T_{\widehat{\lambda}}; \mathrm{wt}, \widehat{\varepsilon}^*_p, \widehat{\varphi}^*_p, \widehat{e}^*_p, \widehat{f}^*_p)$ of type $A^{(1)}_{l-1}$ as follows: $\mathrm{wt}(t_{\widehat{\lambda}}) = \widehat{\lambda}$, and for each $p \in \widehat{I}$, $\widehat{\varepsilon}_p(t_{\widehat{\lambda}}) = \widehat{\varphi}_p(t_{\widehat{\lambda}}) = \widehat{\varepsilon}^*_p(t_{\widehat{\lambda}}) = \widehat{\varphi}^*_p(t_{\widehat{\lambda}}) = -\infty$, $\widehat{e}_p t_{\widehat{\lambda}} = \widehat{f}_p t_{\widehat{\lambda}} = \widehat{e}^*_p t_{\widehat{\lambda}} = \widehat{f}^*_p t_{\widehat{\lambda}} = 0$.

Define a map $\mathcal{T} : \mathcal{B}^{(1)}_{l-1}(\mathbf{z}) \to \mathcal{B}^{(1)}_{l-1}(\mathbf{0}) \otimes T_{-m(\mathbf{z})\delta}$ by

$$\mathbf{a} \quad \mapsto \quad \mathbf{a}(0) \otimes t_{-m(\mathbf{z})\delta}.$$

Theorem 5.3.6 ([LTV])**.** (1) *For an arbitrary* $\mathbf{z} \in \mathcal{Z}$, *each of* $\left(\mathcal{B}^{(1)}_{l-1}(\mathbf{z}); \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p\right)$ *and* $\left(\mathcal{B}^{(1)}_{l-1}(\mathbf{z}); \mathrm{wt}, \widehat{\varepsilon}^*_p, \widehat{\varphi}^*_p, \widehat{e}^*_p, \widehat{f}^*_p\right)$ *is a crystal of type* $A^{(1)}_{l-1}$.

(2) *Each of* $\left(\mathcal{B}^{(1)}_{l-1}(\mathbf{0}); \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p\right)$ *and* $\left(\mathcal{B}^{(1)}_{l-1}(\mathbf{0}); \mathrm{wt}, \widehat{\varepsilon}^*_p, \widehat{\varphi}^*_p, \widehat{e}^*_p, \widehat{f}^*_p\right)$ *is isomorphic to* $B(\infty)$ *of type* $A^{(1)}_{l-1}$.

(3) *The map* $\mathcal{T} : \mathcal{B}^{(1)}_{l-1}(\mathbf{z}) \to \mathcal{B}^{(1)}_{l-1}(\mathbf{0}) \otimes T_{-m(\mathbf{z})\delta}$ *is an isomorphism of crystals with respect to each of the crystal structures above.*

(4) *The decomposition* (5.3.2) *gives a decomposition of* $\mathcal{B}^{(1)}_{l-1}$ *into its connected components. More specifically, we have*

$$\mathcal{B}^{(1)}_{l-1} \cong \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \left(B(\infty) \otimes T_{-m\delta}\right)^{\oplus p(m)},$$

where $p(m)$ *denotes the number of partitions of* $m \geq 0$.

Remark . The crystal structure on $\mathcal{B}^{(1)}_{l-1}$ (or equivalently, on $\mathbf{Seg}(\mathbb{Z}/l\mathbb{Z})$) introduced in [LTV] is $\left(\mathcal{B}^{(1)}_{l-1}; \mathrm{wt}, \widehat{\varepsilon}^*_p, \widehat{\varphi}^*_p, \widehat{e}^*_p, \widehat{f}^*_p\right)$. Therefore, strictly speaking, they proved the statement above only for this crystal structure. However, by a similar method, one can prove the statement for the other crystal structure. Hence we omit the details.

6. BZ data arising from Lusztig data of Infinite size

6.1. **Results on BZ data associated to finite intervals.** In this subsection, we prove some results on BZ data associated to finite intervals, which we need later.

Throughout this subsection, $\mathbf{k}$ denotes a Maya diagram associated to a finite interval. More specifically, let $I = [n+1, n+m]$ be a finite interval and $\mathbf{k} = (k_{n+1} < \cdots < k_{n+u}) \in \mathcal{M}_I^{\times}$. Recall that $k_j \in \widetilde{I} = [n+1, n+m+1]$ for all $n+1 \le j \le n+u$.

Let $d \in \mathbb{Z}_{>0}$, and define

$$I' := [n-d+1, n+m], \quad \mathbf{k}' := [n-d+1, n] \cup \mathbf{k} \in \mathcal{M}_{I'}^{\times},$$

$$I'' := [n+1, n+m+d], \quad \mathbf{k}'' := \mathbf{k} \cup [n+m+2, n+m+d+1] \in \mathcal{M}_{I''}^{\times}.$$

We use the notation above throughout this subsection.

Lemma 6.1.1. *Let $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}$, and suppose that $u \ge N_{\mathbf{a}}$.*
(1) If $k_j = j$ for each $n+1 \le j \le n+N_{\mathbf{a}}$, then we have $M_{\mathbf{k}}(\mathbf{a}^I) = M_{\mathbf{k}'}(\mathbf{a}^{I'})$ for all $d \in \mathbb{Z}_{>0}$.
(2) If $k_j = m-u+j$ for each $n+u-N_{\mathbf{a}}+1 \le j \le n+u$, then we have $M_{\mathbf{k}}(\mathbf{a}^I) = M_{\mathbf{k}''}(\mathbf{a}^{I''})$ for all $d \in \mathbb{Z}_{>0}$.

Proof. We only give a proof of part (1), since part (2) is shown in a similar way. We write

$$\mathbf{k}' = (k_{n-d+1} < \cdots < k_n < k_{n+1} < \cdots < k_{n+u}).$$

Then we have

$$k_j = j \quad \text{for each } n-d+1 \le j \le n+N_{\mathbf{a}}. \tag{6.1.1}$$

The explicit form of $M_{\mathbf{k}}(\mathbf{a}^I)$ is given by

$$M_{\mathbf{k}}(\mathbf{a}^I) = -\sum_{j=n+1}^{n+u} \sum_{i=n+1}^{k_j-1} a_{i,k_j} + \min \left\{ \sum_{n+1 \le p < q \le n+u} a_{c_{p,q}, c_{p,q}+(q-p)} \;\middle|\; \begin{array}{c} C = (c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau} \end{array} \right\}.$$

Let us compute the right-hand side. By (5.1.1) and (6.1.1), we have

$$\begin{aligned} \sum_{j=n+1}^{n+u} \sum_{i=n+1}^{k_j-1} a_{i,k_j} &= \sum_{j=n+1}^{n+N_{\mathbf{a}}} \sum_{i=n+1}^{k_j-1} a_{i,k_j} + \sum_{j=n+N_{\mathbf{a}}+1}^{n+u} \sum_{i=n+1}^{k_j-1} a_{i,k_j} \\ &= \sum_{n+1 \le i < j \le n+N_{\mathbf{a}}} a_{i,j} + \sum_{j=n+N_{\mathbf{a}}+1}^{n+u} \sum_{i=k_j-N_{\mathbf{a}}+1}^{k_j-1} a_{i,k_j}. \end{aligned}$$

Now, let $C = (c_{p,q})_{n+1 \le p < q \le n+u}$ be a $\mathbf{k}^I$-tableau. Because of (6.1.1) and the monotonicity condition for C , we have $c_{p,q} = p$ for $n+1 \le p < q \le n+N_{\mathbf{a}}$. Therefore, by (5.1.1), we compute:

$$\begin{aligned} &\sum_{n+1 \le p < q \le n+u} a_{c_{p,q}, c_{p,q}+(q-p)} \\ &\quad = \sum_{n+1 \le p < q \le n+N_{\mathbf{a}}} a_{c_{p,q}, c_{p,q}+(q-p)} + \sum_{q=n+N_{\mathbf{a}}+1}^{n+u} \sum_{p=n+1}^{q-1} a_{c_{p,q}, c_{p,q}+(q-p)} \\ &\quad = \sum_{n+1 \le p < q \le n+N_{\mathbf{a}}} a_{p,q} + \sum_{q=n+N_{\mathbf{a}}+1}^{n+u} \sum_{p=q-N_{\mathbf{a}}}^{q-1} a_{c_{p,q}, c_{p,q}+(q-p)}. \end{aligned}$$

Consequently, we deduce that

$$M_{\mathbf{k}}(\mathbf{a}^{I}) = -\sum_{j=n+N_{\mathbf{a}}+1}^{n+u}\sum_{i=k_j-N_{\mathbf{a}}+1}^{k_j-1} a_{i,k_j} + \min\left\{ \sum_{q=n+N_{\mathbf{a}}+1}^{n+u}\sum_{p=q-N_{\mathbf{a}}}^{q-1} a_{c_{p,q},c_{p,q}+(q-p)} \;\middle|\; \begin{array}{c} C=(c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau} \end{array} \right\}.$$

By a similar computation, we deduce that

$$M_{\mathbf{k}'}(\mathbf{a}^{I'}) = -\sum_{j=n+N_{\mathbf{a}}+1}^{n+u}\sum_{i=k_j-N_{\mathbf{a}}+1}^{k_j-1} a_{i,k_j} + \min\left\{ \sum_{q=n+N_{\mathbf{a}}+1}^{n+u}\sum_{p=q-N_{\mathbf{a}}}^{q-1} a_{c'_{p,q},c'_{p,q}+(q-p)} \;\middle|\; \begin{array}{c} C'=(c'_{p,q}) \text{ is} \\ \text{a } \mathbf{k}'\text{-tableau} \end{array} \right\}.$$

Since it is obvious that

$$\min\left\{ \sum_{q=n+N_{\mathbf{a}}+1}^{n+u}\sum_{p=q-N_{\mathbf{a}}}^{q-1} a_{c_{p,q},c_{p,q}+(q-p)} \;\middle|\; \begin{array}{c} C=(c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau} \end{array} \right\} = \min\left\{ \sum_{q=n+N_{\mathbf{a}}+1}^{n+u}\sum_{p=q-N_{\mathbf{a}}}^{q-1} a_{c'_{p,q},c'_{p,q}+(q-p)} \;\middle|\; \begin{array}{c} C'=(c'_{p,q}) \text{ is} \\ \text{a } \mathbf{k}'\text{-tableau} \end{array} \right\},$$

we obtain the desired equality. □

Note that $\mathbf{k} \in \mathcal{M}_I^{\times}$ can be naturally regarded as an element of $\mathcal{M}_{I'}^{\times}$ and of $\mathcal{M}_{I''}^{\times}$. The next lemma follows easily from the definitions and (5.1.1).

Lemma 6.1.2. *Let $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}$.*
(1) *If $k_{n+1} - n \geq N_{\mathbf{a}}$, then $M_{\mathbf{k}}(\mathbf{a}^{I}) = M_{\mathbf{k}}(\mathbf{a}^{I'})$ for all $d \in \mathbb{Z}_{>0}$.*
(2) *For all $d \in \mathbb{Z}_{>0}$, we have $M_{\mathbf{k}}(\mathbf{a}^{I}) = M_{\mathbf{k}}(\mathbf{a}^{I''})$.*

Lemma 6.1.3. *Let $\mathbf{a} \in \mathcal{B}_{l-1}^{(1),ap}$. Assume that the following condition is satisfied:*

$$k_{n+1} - n \geq (2l+1)N_{\mathbf{a}}. \tag{6.1.2}$$

Then, we have $M_{\mathbf{k}}(\mathbf{a}^{I}) = M_{\mathbf{k}'}(\mathbf{a}^{I'})$ for all $d \in \mathbb{Z}_{>0}$.

Proof. We use the same notation as in the proof of Lemma 6.1.1:

$$\mathbf{k}' = (k_{n-d+1} < \cdots < k_n < k_{n+1} < \cdots < k_{n+u}).$$

In this case, we have

$$k_j = j \quad \text{for each } n-d+1 \leq j \leq n. \tag{6.1.3}$$

Since $k_{n+1} - n \geq (2l+1)N_{\mathbf{a}} > N_{\mathbf{a}}$, we obtain

$$M_{\mathbf{k}}(\mathbf{a}^{I}) = -\sum_{j=n+1}^{n+u}\sum_{i=k_j-N_{\mathbf{a}}+1}^{k_j-1} a_{i,k_j} + \min\left\{ \sum_{q=n+2}^{n+u}\sum_{p=n+1}^{q-1} a_{c_{p,q},c_{p,q}+(q-p)} \;\middle|\; \begin{array}{c} C=(c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau} \end{array} \right\}.$$

By (6.1.3) and an argument similar to the one in the proof of Lemma 6.1.1, we deduce that

$$M_{\mathbf{k}'}(\mathbf{a}^{I'}) = -\sum_{j=n+1}^{n+u}\ \sum_{i=k_j-N_{\mathbf{a}}+1}^{k_j-1} a_{i,k_j} \\ + \min\left\{ \sum_{q=n+1}^{n+u}\ \sum_{p=n-d+1}^{q-1} a_{c'_{p,q},c'_{p,q}+(q-p)} \ \middle|\ \begin{array}{c} C'=(c'_{p,q}) \text{ is} \\ \text{a } \mathbf{k}'\text{-tableau} \end{array} \right\}.$$

Let us write

$$S := \min\left\{ \sum_{q=n+2}^{n+u}\ \sum_{p=n+1}^{q-1} a_{c_{p,q},c_{p,q}+(q-p)} \ \middle|\ \begin{array}{c} C=(c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau} \end{array} \right\},$$

$$S' := \min\left\{ \sum_{q=n+1}^{n+u}\ \sum_{p=n-d+1}^{q-1} a_{c'_{p,q},c'_{p,q}+(q-p)} \ \middle|\ \begin{array}{c} C'=(c'_{p,q}) \text{ is} \\ \text{a } \mathbf{k}'\text{-tableau} \end{array} \right\}.$$

By the computation above, it suffices to show that $S = S'$ under the assumption (6.1.2). Observe that S' can be rewritten as

$$S' = \min\left\{ \sum_{q=n+2}^{n+u}\ \sum_{p=n+1}^{q-1} a_{c'_{p,q},c'_{p,q}+(q-p)} + \sum_{q=n+1}^{n+u}\ \sum_{p=n-d+1}^{n} a_{c'_{p,q},c'_{p,q}+(q-p)} \ \middle|\ \begin{array}{c} C'=(c'_{p,q}) \text{ is} \\ \text{a } \mathbf{k}'\text{-tableau} \end{array} \right\}.$$

For each $\mathbf{k}'$-tableau $C' = (c'_{p,q})_{n-d+1\le p\le q\le n+u}$, the subtableau $(c'_{p,q})_{n+1\le p\le q\le n+u}$ of C' is always a $\mathbf{k}$-tableau. Therefore, what we need to show is the following: if the assumption (6.1.2) holds, then there exists a $\mathbf{k}'$-tableau $C' = (c'_{p,q})_{n-d+1\le p\le q\le n+u}$ such that

$$\sum_{q=n+2}^{n+u}\ \sum_{p=n+1}^{q-1} a_{c'_{p,q},c'_{p,q}+(q-p)} = S \quad \text{and} \quad \sum_{q=n+1}^{n+u}\ \sum_{p=n-d+1}^{n} a_{c'_{p,q},c'_{p,q}+(q-p)} = 0. \tag{6.1.4}$$

In the following, we will construct such a C'.

For $1 \le v \le N_{\mathbf{a}}$, consider the l-tuple of nonnegative integers:

$$\mathcal{A}_v := \left\{ a_{i,i+N_{\mathbf{a}}+1-v} \ \middle|\ n+v \le i \le n+v+l-1 \right\}.$$

Note that, by assumption (6.1.2), the following inequalities hold:

$$n-d+1 \le i < i+N_{\mathbf{a}}+1-v \le n+m \\ \text{for all } 1 \le v \le N_{\mathbf{a}} \text{ and } n+v \le i \le n+v+l-1.$$

By the aperiodicity condition for $\mathbf{a}$, there exists at least one 0 in $\mathcal{A}_v$. Set

$$b_v := \min\left\{ i \ \middle|\ a_{i,i+N_{\mathbf{a}}+1-v} \in \mathcal{A}_v \text{ and } a_{i,i+N_{\mathbf{a}}+1-v} = 0 \right\}.$$

Then it follows that

$$n+v \le b_v \le n+v+l-1. \tag{6.1.5}$$

Set $N_0 := \max\{N_{\mathbf{a}}, u, d\}$, and introduce an $(N_0 \times N_0)$-matrix $X = (x_{s,t})$, where the indices run over $n-N_0+1 \le s \le n$ and $n+1 \le t \le n+N_0$, by

$$x_{s,t} := \begin{cases} s & \text{if } n-N_0+1 \le s \le n-N_{\mathbf{a}}, \\ b_1 + 2(s-n+N_{\mathbf{a}}-1)l & \text{if } n-N_{\mathbf{a}}+1 \le s \le n,\ s+N_{\mathbf{a}} \le t, \\ b_{s-t+N_{\mathbf{a}}+1} + (s+t-2n+N_{\mathbf{a}}-2)l & \text{if } n-N_{\mathbf{a}}+1 \le s \le n,\ s+N_{\mathbf{a}} > t. \end{cases}$$

Claim. (1) *The entries of X satisfy the (usual) monotonicity conditions. Namely, for all s and t, $x_{s,t} \leq x_{s,t+1}$ and $x_{s,t} < x_{s+1,t}$.*
(2) *For all $n - N_0 + 1 \leq s \leq n$, we have $s \leq x_{s,n+1}$.*

Proof of the claim. If $n-N_0+1 \leq s \leq n-N_{\mathbf{a}}$, or if $n-N_{\mathbf{a}}+1 \leq s \leq n$ and $s+N_{\mathbf{a}} \leq t$, then the monotonicity conditions are obviously satisfied. Suppose that $n - N_{\mathbf{a}} + 1 \leq s \leq n$ and $s+N_{\mathbf{a}} > t$. By (6.1.5), we have $b_{s-t+N_{\mathbf{a}}} \geq n+s-t+N_{\mathbf{a}}$ and $b_{s-t+N_{\mathbf{a}}+1} \leq n+s-t+N_{\mathbf{a}}+l$. Therefore, we see that

$$\begin{aligned}x_{s,t+1} - x_{s,t} &= b_{s-t+N_{\mathbf{a}}} - b_{s-t+\mathbf{a}+1} + l\\&\geq (n+s-t+N_{\mathbf{a}}) - (n+s-t+N_{\mathbf{a}}+l) + l\\&\geq 0.\end{aligned}$$

By a similar computation, we obtain $x_{s,t} < x_{s+1,t}$. This proves part (1).

We prove part (2). If $n - N_0 + 1 \leq s \leq n - N_{\mathbf{a}}$, then the assertion is obvious. Let $n - N_{\mathbf{a}} + 1 \leq s \leq n$. By (6.1.5), we see that

$$\begin{aligned}x_{s,n+1} - s &= b_{s-(n+1)+N_{\mathbf{a}}+1} + (s + (n+1) - 2n + N_{\mathbf{a}} - 2)l - s\\&\geq n + (s - n + N_{\mathbf{a}}) + (s - n + N_{\mathbf{a}} - 1)l - s\\&= N_{\mathbf{a}} + (s - n + N_{\mathbf{a}} - 1)l\\&\geq 0.\end{aligned}$$

This proves part (2). □

Now let us construct a $\mathbf{k}'$-tableau $C' = (c'_{p,q})_{n-d+1 \leq p \leq q \leq n+u}$ satisfying (6.1.4).

First, we remark that $c'_{p,q}$ must be equal to p for $n - d + 1 \leq p \leq q \leq n$ by (6.1.3). Second, for $n - d - 1 \leq p \leq n$ and $n + 1 \leq q \leq n + u$, we set $c'_{p,q} := x_{p,q}$. Here, $x_{p,q}$ is the (p, q)-th entry of the matrix X constructed above. By the claim above, the monotonicity conditions are satisfied for all $n - d + 1 \leq p \leq n$ and $p \leq q \leq n + u$. Finally, define $c'_{p,q}$ for $n+1 \leq p \leq q \leq n+u$ as follows. Let $C = (c_{p,q})_{n+1 \leq p \leq q \leq n+u}$ be a $\mathbf{k}$-tableau such that $\sum_{q=n+2}^{n+u} \sum_{p=n+1}^{q-1} a_{c_{p,q},c_{p,q}+(q-p)} = S$. We set $c'_{p,q} = c_{p,q}$ for $n + 1 \leq p \leq q \leq n + u$. The upper triangular matrix $C' = (c'_{p,q})_{n-d+1 \leq p \leq q \leq n+u}$ thus obtained is a $\mathbf{k}'$-tableau. Indeed, by the monotonicity condition for X, the largest matrix entry is $x_{n,n+N_0} = b_1+2(N_{\mathbf{a}}-1)l$. By (6.1.2) and (6.1.5), we see that

$$\begin{aligned}k_{n+1} - x_{n,n+N_0} &\geq k_{n+1} - (n + l) - 2(N_{\mathbf{a}} - 1)l\\&= (k_{n+1} - n) - (2l + 1)N_{\mathbf{a}} + l + N_{\mathbf{a}}\\&> 0.\end{aligned}$$

Hence it follows that $c'_{n,q} \leq x_{n,n+N_0} < k_{n+1} \leq c'_{n+1,q}$ for all $n + 1 \leq q \leq n + u$. Therefore, the monotonicity conditions are satisfied for all $n - d - 1 \leq p \leq q \leq n + u$.

Moreover, by the construction, the condition (6.1.4) is automatically satisfied. Thus, the lemma is proved. □

6.2. **Collections of integers arising from Lusztig data associated to $\mathbb{Z}$.** Let us return to the infinite case.

Corollary 6.2.1. *Let $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}$ and $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$. Then, there exists a finite interval I_0 such that* (i) *$\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I_0)$, and* (ii) *for every finite interval J containing I_0, $M_{\mathrm{res}_J(\mathbf{k})}(\mathbf{a}^J) = M_{\mathrm{res}_{I_0}(\mathbf{k})}(\mathbf{a}^{I_0})$.*

Proof. Take a finite interval $I = [n+1, n+m]$ such that $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I)$. Note that $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(J)$ for all $J \supset I$. Set $I_0 := [n - N_{\mathbf{a}} + 1, n + m]$. Then, by Lemma 6.1.1 (1), I_0 has the desired properties. $\square$

For given $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}$ and $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$, take a finite interval I_0 of Corollary 6.2.1 and define $M_{\mathbf{k}}(\mathbf{a}) := M_{\mathrm{res}_{I_0}(\mathbf{k})}(\mathbf{a}^{I_0})$. Note that this definition does not depend on the choice of I_0 by the corollary above. Set $\mathbf{M}(\mathbf{a}) := (M_{\mathbf{k}}(\mathbf{a}))_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}}$, and define a map $\Phi_{\mathbb{Z}}$ from $\mathcal{B}_{\mathbb{Z}}$ to the set of all collections of integers indexed by $\mathcal{M}_{\mathbb{Z}}$, by the assignment $\mathbf{a} \mapsto \mathbf{M}(\mathbf{a})$.

We remark that $\Phi_{\mathbb{Z}}$ is not injective. Indeed, take $\mathbf{0} = (0, 0, \ldots) \in \mathcal{Z}$ and consider $\mathbf{a_0} \in \mathcal{B}_{\mathbb{Z}}$. Then we have $\Phi_{\mathbb{Z}}(\mathbf{a_0}) = \mathbf{O}^* \in \mathcal{B}Z^e_{\mathbb{Z}}$. Also, take $\mathbf{1} = (1, 0, 0, \ldots) \in \mathcal{Z}$ and consider $\mathbf{a_1} \in \mathcal{B}_{\mathbb{Z}}$; namely, $\mathbf{a_1} = (a_{i,j})$ is given by $a_{i,j} = \delta_{j-i,1}$ for each $(i, j) \in \Delta^+_{\mathbb{Z}}$. We may take $N_{\mathbf{a_1}} = 2$. Let $\mathbf{k} = \{k_j \mid j \in \mathbb{Z}_{\le r}\} \in \mathcal{M}^{(r)}_{\mathbb{Z}}$ and $I_0 = [n_0 + 1, n_0 + m_0]$. Then we compute:

$$\begin{aligned} M_{\mathbf{k}}(\mathbf{a_1}) &= M_{res_{I_0}(\mathbf{k}^{I_0})}\left((\mathbf{a_1})^{I_0}\right) \\ &= -\sum_{j=n_0+N_{\mathbf{a}}+1}^{r} \sum_{i=k_j-1}^{k_j-1} \delta_{k_j-i,1} + \min\left\{ \sum_{q=n_0+N_{\mathbf{a}}+1}^{r} \sum_{p=q-2}^{q-1} \delta_{q-p,1} \;\middle|\; \begin{array}{c} C = (c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}^{I_0}\text{-tableau} \end{array} \right\} \\ &= -(r - n_0 - N_{\mathbf{a}}) + (r - n_0 - N_{\mathbf{a}}) \\ &= 0. \end{aligned}$$

Therefore, we conclude that $\Phi_{\mathbb{Z}}(\mathbf{a_1}) = \mathbf{O}^* = \Phi_{\mathbb{Z}}(\mathbf{a_0})$.

By a similar computation, we have the following.

Lemma 6.2.2. *Let $\mathbf{a_z}$ be a maximal element in $\mathcal{B}^{(1)}_{l-1}(\mathbf{z})$. Then we have $\Phi_{\mathbb{Z}}(\mathbf{a_z}) = \mathbf{O}^*$.*

6.3. **BZ data arising from aperiodic Lusztig data.** The aim of this subsection is to prove the following proposition.

Proposition 6.3.1. *For each $\mathbf{a} \in \mathcal{B}^{(1),ap}_{l-1} = \mathcal{B}^{(1)}_{l-1}(\mathbf{0})$, the collection $\mathbf{M}(\mathbf{a}) = (M_{\mathbf{k}}(\mathbf{a}))_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}}$ is an e-BZ datum in the sense of Definition 3.3.2. In other words, the restriction of $\Phi_{\mathbb{Z}}$ to $\mathcal{B}^{(1),ap}_{l-1}$ defines a map $\Phi_{\mathbb{Z}} : \mathcal{B}^{(1),ap}_{l-1} \to \mathcal{B}Z^e_{\mathbb{Z}}$.*

In order to prove the proposition, we need the next lemma.

Lemma 6.3.2. *Let $\mathbf{a} \in \mathcal{B}^{(1),ap}_{l-1}$ and $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$. Then, there exists a finite interval I in $\mathbb{Z}$ such that, for every finite interval $J \supset I$, $M_{\Omega_J(\mathbf{k})}(\mathbf{a}) = M_{\Omega_I(\mathbf{k})}(\mathbf{a})$.*

Proof. We may assume that $\mathbf{k}$ is a Maya diagram of charge r, and write $\mathbf{k} = \{k_j \mid j \in \mathbb{Z}_{\le r}\}$. Define $j_0 := \max\{j \in \mathbb{Z}_{\le r} \mid k_j = j\}$, and set

$$I_1 := [j_0 + 1, k_r - 1], \quad \text{and} \quad I := [j_0 - (2l+1)N_{\mathbf{a}} + 1, k_r + N_{\mathbf{a}} - 1].$$

Then we have $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(I_1) \subset \mathcal{M}_{\mathbb{Z}}(I)$. Hence we can consider $\Omega_I(\mathbf{k}) \in \mathcal{M}_{\mathbb{Z}}(I)$. Now let us recall the definition of $M_{\Omega_I(\mathbf{k})}(\mathbf{a})$; we set $I_0 := [j_0 - (2l+2)N_{\mathbf{a}} + 1, k_r + N_{\mathbf{a}} - 1]$, and define

$$M_{\Omega_I(\mathbf{k})}(\mathbf{a}) = M_{\mathrm{res}_{I_0}(\Omega_I(\mathbf{k}))}(\mathbf{a}^{I_0}).$$

Note that the explicit forms of $\mathrm{res}_I\big(\Omega_I(\mathbf{k})\big)$ and $\mathrm{res}_{I_0}\big(\Omega_I(\mathbf{k})\big)$ are given as:

$$\begin{aligned} \mathrm{res}_I\big(\Omega_I(\mathbf{k})\big) &= \left(\widetilde{I}_1 \setminus \mathrm{res}_{I_1}\right) \cup [k_r + 1, k_r + N_{\mathbf{a}}] \quad \text{and} \\ \mathrm{res}_{I_0}\big(\Omega_I(\mathbf{k})\big) &= [j_0 - (2l+2)N_{\mathbf{a}} + 1, j_0 - (2l+1)N_{\mathbf{a}}] \\ &\qquad \cup \left(\widetilde{I}_1 \setminus \mathrm{res}_{I_1}\right) \cup [k_r + 1, k_r + N_{\mathbf{a}}], \end{aligned}$$

respectively. Since $(j_0+1)-(j_0-(2l+1)N_{\mathbf{a}}) > (2l+1)N_{\mathbf{a}}$, we obtain

$$M_{\Omega_I(\mathbf{k})}(\mathbf{a}) = M_{\mathrm{res}_{I_0}(\Omega_I(\mathbf{k}))}(\mathbf{a}^{I_0}) = M_{\mathrm{res}_I(\Omega_I(\mathbf{k}))}(\mathbf{a}^{I}) \tag{6.3.1}$$

by Lemma 6.1.3.

Let J be a finite interval such that $J \supset I$. Write $J = [j_0-(2l+1)N_{\mathbf{a}}+1-d_1, k_r+N_{\mathbf{a}}-1+d_2]$ for some $d_1, d_2 \in \mathbb{Z}_{\geq 0}$, and set $J_0 := [j_0-(2l+2)N_{\mathbf{a}}+1-d_1, k_r+N_{\mathbf{a}}-1+d_2]$. Then we have

$$M_{\Omega_J(\mathbf{k})}(\mathbf{a}) = M_{\mathrm{res}_{J_0}(\Omega_J(\mathbf{k}))}(\mathbf{a}^{J_0}).$$

The explicit form of $\mathrm{res}_{J_0}\big(\Omega_J(\mathbf{k})\big)$ is given as:

$$\begin{aligned}\mathrm{res}_{J_0}\big(\Omega_J(\mathbf{k})\big) = [j_0-(2l+2)N_{\mathbf{a}}+1-d_1, j_0-(2l+1)N_{\mathbf{a}}-d_1]\\ \cup\Big(\widetilde{I}_1\setminus \mathrm{res}_{I_1}\Big)\cup[k_r+1, k_r+N_{\mathbf{a}}+d_2].\end{aligned}$$

Since $(j_0+1)-(j_0-(2l+1)N_{\mathbf{a}}-d_1) > (2l+1)N_{\mathbf{a}} > N_{\mathbf{a}}$ and $[k_r+1, k_r+N_{\mathbf{a}}] \subset [k_r+1, k_r+N_{\mathbf{a}}+d_2]$, we deduce that

$$M_{\Omega_J(\mathbf{k})}(\mathbf{a}) = M_{\mathrm{res}_{J_0}(\Omega_J(\mathbf{k}))}(\mathbf{a}^{J_0}) = M_{\mathrm{res}_I(\Omega_I(\mathbf{k}))}(\mathbf{a}^{I})$$

by part (2) of Lemma 6.1.1, part (1) of Lemma 6.1.2, and Lemma 6.1.3. Thus, we conclude that $M_{\Omega_J(\mathbf{k})}(\mathbf{a}) = M_{\Omega_I(\mathbf{k})}(\mathbf{a})$. □

Proof of Proposition 6.3.1. Let us check condition (2-a) in Definition 3.3.2. Take a finite interval K, and consider a finite collection $\mathbf{M}(\mathbf{a})_K = (M_{\mathbf{k}}(\mathbf{a}))_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}(K)}$ of integers. It suffices to prove that $\mathbf{M}(\mathbf{a})_K$ is an element of $\mathcal{B}Z^e_K$ under the identification $\mathcal{M}_{\mathbb{Z}}(K) \cong \mathcal{M}^{\times}_K$. Since $\mathcal{M}_{\mathbb{Z}}(K)$ is a finite set, there exists a finite interval I such that $I \supset K$ and $M_{\mathbf{k}}(\mathbf{a}) = M_{\mathbf{k}^I}(\mathbf{a}^I)$ for all $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(K)$. Hence the condition (2-a) is satisfied by Theorem 2.4.3.

It remains to check the condition (2-b), which is clear from Lemma 6.3.2. □

The following corollary is easily obtained.

Corollary 6.3.3. *For each $\mathbf{a} \in \mathcal{B}^{(1),ap}_{l-1}$, $\Phi_{\mathbb{Z}}(\mathbf{a})$ is an element of $(\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}$.*

6.4. **Comparison theorem.** In the previous subsection, we have constructed the map $\Phi_{\mathbb{Z}} : \mathcal{B}^{(1),ap}_{l-1} \to (\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}$ by the assignment $\mathbf{a} \mapsto \mathbf{M}(\mathbf{a})$. In this subsection, we show that $\Phi_{\mathbb{Z}}$ is a morphism of crystals.

Lemma 6.4.1. *Let $\mathbf{a} \in \mathcal{B}^{(1),ap}_{l-1}$. Then,*

(1) $\mathrm{wt}\big(\mathbf{M}(\mathbf{a})\big) = \mathrm{wt}(\mathbf{a})$.

(2) $\widehat{\varepsilon}^*_p\big(\mathbf{M}(\mathbf{a})\big) = \widehat{\varepsilon}^*_p(\mathbf{a})$ *for all $p \in \widehat{I}$.*

(3) *For all $p \in \mathbb{Z}$, each of $\widetilde{e}^*_p$ and $\widetilde{f}^*_p$ commutes with $\Phi_{\mathbb{Z}}$.*

Proof. Recall the definition of the weight of $\mathbf{M}(\mathbf{a}) \in (\mathcal{B}Z^e_{\mathbb{Z}})^{\sigma}$:

$$\mathrm{wt}\big(\mathbf{M}(\mathbf{a})\big) = \sum_{p\in\widehat{I}} \Theta\big(\mathbf{M}^*(\mathbf{a})\big)_{\mathbf{k}(\Lambda_p)}\widehat{\alpha}_p.$$

Here we write $\mathbf{M}^*(\mathbf{a}) = \big(\mathbf{M}(\mathbf{a})\big)^*$. Take a sufficiently large finite interval J. Then, for all $p \in \widehat{I}$, we have

$$\Theta\big(\mathbf{M}^*(\mathbf{a})\big)_{\mathbf{k}(\Lambda_p)} = M^*_{\Omega^c_J((\mathbf{k}(\Lambda_p))^c)}(\mathbf{a}) = M_{(\Omega^c_J((\mathbf{k}(\Lambda_p))^c))^c}(\mathbf{a}) = M_{\Omega_J(\mathbf{k}(\Lambda_p))}(\mathbf{a}),$$

where we use (3.3.3) for the last equality. Since J is sufficiently large, we obtain the following equalities by the same argument as in the proof of (6.3.1):

$$M_{\Omega_J(\mathbf{k}(\Lambda_p))}(\mathbf{a}) = M_{\mathrm{res}_J(\Omega_J(\mathbf{k}(\Lambda_p)))}(\mathbf{a}^J) \quad \text{for all } p \in \widehat{I}.$$

Write $J = [n+1, n+m]$. Note that $n+1 < p < n+m$ by the construction. Hence we have

$$\mathrm{res}_J(\Omega_J(\mathbf{k}(\Lambda_p))) = [p+1, n+m+1] \in \mathcal{M}_J^{\times}.$$

Since J is sufficiently large, we may assume that $n+m-p > N_{\mathbf{a}}$. Therefore, we see that

$$\Theta\big(\mathbf{M}^*(\mathbf{a})\big)_{\mathbf{k}(\Lambda_p)} = M_{[p+1,n+m+1]}(\mathbf{a}^J) = \sum_{s \le p} \sum_{t \ge p+1} a_{s,t}.$$

The right-hand side is exactly $r_p(\mathbf{a})$. This proves part (1).

Let us prove part (2). Since $\widehat{\varepsilon}_p^*\big(\mathbf{M}(\mathbf{a})\big) = \varepsilon_p^*\big(\mathbf{M}(\mathbf{a})\big)$ and $\widehat{\varepsilon}_p^*(\mathbf{a}) = \varepsilon_p^*(\mathbf{a})$, it suffices to show that

$$\varepsilon_p^*\big(\mathbf{M}(\mathbf{a})\big) = \varepsilon_p^*(\mathbf{a}) \quad \text{for all } p \in \mathbb{Z}. \tag{6.4.1}$$

By the definitions of $\varepsilon_p^*\big(\mathbf{M}(\mathbf{a})\big)$ and $\varepsilon_p^*(\mathbf{a})$, this is equivalent to:

$$\Theta\big(\mathbf{M}^*(\mathbf{a})\big)_{\mathbf{k}(\Lambda_p)} + \Theta\big(\mathbf{M}^*(\mathbf{a})\big)_{\mathbf{k}(\sigma_p\Lambda_p)} - \Theta\big(\mathbf{M}^*(\mathbf{a})\big)_{\mathbf{k}(\Lambda_{p+1})} - \Theta\big(\mathbf{M}^*(\mathbf{a})\big)_{\mathbf{k}(\Lambda_{p-1})} = -\varepsilon_p(\mathbf{a}^J)$$

for a sufficiently large finite interval J. However, this equality can be obtained by a computation similar to the above for part (1). This proves part (2).

For part (3), we only give a proof of $\widetilde{e}_p^* \circ \Phi_{\mathbb{Z}} = \Phi_{\mathbb{Z}} \circ \widetilde{e}_p^*$, since the other one follows similarly. By (6.4.1), $\widetilde{e}_p^*\mathbf{M}(\mathbf{a}) = 0$ if and only if $\widetilde{e}_p^*(\mathbf{a}) = 0$. Therefore, it suffices to show the commutativity under the assumption that $\widetilde{e}_p^*\mathbf{M}(\mathbf{a}) \in (\mathcal{B}Z_{\mathbb{Z}}^{e})^{\sigma}$ (or equivalently, $\widetilde{e}_p^*\mathbf{a} \in \mathcal{B}_{l-1}^{(1),ap}$).

Fix a Maya diagram $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$. Then, there exists a finite interval J such that $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}(J)$ and such that the $\mathbf{k}$-component of $\widetilde{e}_p^*\mathbf{M}(\mathbf{a})$ is given by

$$\big(\widetilde{e}_p^*\mathbf{M}(\mathbf{a})\big)_{\mathbf{k}} = \big(\widetilde{e}_p^*\big(\mathbf{M}(\mathbf{a})_J\big)\big)_{\mathrm{res}_J(\mathbf{k})}. \tag{6.4.2}$$

Here, $\mathbf{M}(\mathbf{a})_J = \big(M_{\mathbf{m}}(\mathbf{a})\big)_{\mathbf{m} \in \mathcal{M}_{\mathbb{Z}}(J)}$. Since $\mathcal{M}_{\mathbb{Z}}(J)$ is a finite set, there exists a sufficiently large interval K such that $K \supset J$ and such that, for all $\mathbf{m} \in \mathcal{M}_{\mathbb{Z}}(J)$,

$$M_{\mathbf{m}}(\mathbf{a}) = M_{\mathrm{res}_K(\mathbf{m})}\big(\mathbf{a}^K\big), \tag{6.4.3}$$

$$M_{\mathbf{m}}(\widetilde{e}_p^*\mathbf{a}) = M_{\mathrm{res}_K(\mathbf{m})}\left((\widetilde{e}_p^*\mathbf{a})^K\right). \tag{6.4.4}$$

Since K is sufficiently large, we may assume that

$$(\widetilde{e}_p^*\mathbf{a})^K = \widetilde{e}_p^*\left(\mathbf{a}^K\right). \tag{6.4.5}$$

Consider a collection $\mathbf{M}' := \big(M_{\mathrm{res}_K(\mathbf{n})}\big(\mathbf{a}^K\big)\big)_{\mathbf{n} \in \mathcal{M}_{\mathbb{Z}}(K)}$. It is the image of the Lusztig datum $\mathbf{a}^K$ associated to the finite interval K under the map $\Phi_K : \mathcal{B}_K \to \mathcal{B}Z_K^{e}$ constructed in Subsection 2.4.1. By Theorem 2.4.3 and (6.4.5), we see that

$$\big(\widetilde{e}_p^*\mathbf{M}'\big)_{\mathrm{res}_K(\mathbf{n})} = M_{\mathrm{res}_K(\mathbf{n})}\big(\widetilde{e}_p^*\big(\mathbf{a}^K\big)\big) = M_{\mathrm{res}_K(\mathbf{n})}\big((\widetilde{e}_p^*\mathbf{a})^K\big) \tag{6.4.6}$$

for all $\mathbf{n} \in \mathcal{M}_{\mathbb{Z}}(K)$. Note that the equalities above hold for all $\mathbf{m} \in \mathcal{M}_{\mathbb{Z}}(J)$ since $\mathcal{M}_{\mathbb{Z}}(J) \subset \mathcal{M}_{\mathbb{Z}}(K)$.

Set $\mathbf{M}'' := (\mathbf{M}')_J = \big(M_{\mathrm{res}_K(\mathbf{m})}\big(\mathbf{a}^K\big)\big)_{\mathbf{m} \in \mathcal{M}_{\mathbb{Z}}(J)}$. Then it follows that

$$\big(\widetilde{e}_p^*\mathbf{M}''\big)_{\mathrm{res}_K(\mathbf{m})} = \big(\widetilde{e}_p^*\mathbf{M}'\big)_{\mathrm{res}_K(\mathbf{m})} \quad \text{for all } \mathbf{m} \in \mathcal{M}_{\mathbb{Z}}(J).$$

Therefore, by (6.4.4), (6.4.6), and these equalities, we obtain

$$M_{\mathbf{m}}(\widetilde{e}_p^*\mathbf{a}) = \left(\widetilde{e}_p^*\mathbf{M}''\right)_{\mathrm{res}_K(\mathbf{m})}.$$

In particular, we have

$$M_{\mathbf{k}}(\widetilde{e}_p^*\mathbf{a}) = \left(\widetilde{e}_p^*\mathbf{M}''\right)_{\mathrm{res}_K(\mathbf{k})}.$$

Also, by (6.4.3), it follows that

$$\mathbf{M}'' = \left(M_{\mathrm{res}_K(\mathbf{m})}(\mathbf{a}^K)\right)_{\mathbf{m}\in\mathcal{M}_{\mathbb{Z}}(J)} = \left(\mathbf{M}_{\mathbf{m}}(\mathbf{a})\right)_{\mathbf{m}\in\mathcal{M}_{\mathbb{Z}}(J)},$$

from which we deduce that $\mathbf{M}'' = \mathbf{M}(\mathbf{a})_J$. Consequently, by (6.4.2), we obtain

$$\left(\widetilde{e}_p^*\mathbf{M}(\mathbf{a})\right)_{\mathbf{k}} = M_{\mathbf{k}}(\widetilde{e}_p^*\mathbf{a}).$$

This proves part (3). □

Proposition 6.4.2. *The map $\Phi_{\mathbb{Z}} : \mathcal{B}_{l-1}^{(1),ap} \to (\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma$ gives rise to a strict morphism of crystals form $\left(\mathcal{B}_{l-1}^{(1),ap}; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*\right)$ to $\left((\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*\right)$.*

Proof. By part (1), (2) of Lemma 6.4.1, it suffices to show that

$$\widehat{e}_p^*\mathbf{M}(\mathbf{a}) = \mathbf{M}(\widehat{e}_p^*\mathbf{a}) \quad \text{and} \quad \widehat{f}_p^*\mathbf{M}(\mathbf{a}) = \mathbf{M}(\widehat{f}_p^*\mathbf{a})$$

for all $p \in \widehat{I}$ and $\mathbf{a} \in \mathcal{B}_{l-1}^{(1),ap}$. Here it is understood that $\mathbf{M}(0) = 0$.

In the following, we only give a proof of the first equality. Note that we may assume that $\widehat{e}_p^*\mathbf{M}(\mathbf{a}) \neq 0$ (or equivalently, $\widehat{e}_p^*\mathbf{a} \neq 0$).

Let $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$. By the definitions, the $\mathbf{k}$-component of $\widehat{e}_p^*\mathbf{M}(\mathbf{a})$ is given as:

$$\left(\widehat{e}_p^*\mathbf{M}(\mathbf{a})\right)_{\mathbf{k}} = \left(\widetilde{e}_{L(\mathbf{k}^c,p)}^*\mathbf{M}(\mathbf{a})\right)_{\mathbf{k}},$$

where $L(\mathbf{k}^c, p) = \{q \in p + l\mathbb{Z} \mid q \notin \mathbf{k} \text{ and } q+1 \in \mathbf{k}\}$ and $\widetilde{e}_{L(\mathbf{k}^c,p)}^* = \prod_{q\in L(\mathbf{k}^c,p)} \widetilde{e}_q^*$. Therefore, by the definition of $\widehat{e}_p^*$ on $\mathcal{B}_{l-1}^{(1)}$ and Lemma 6.4.1 (3), it suffices to verify that

$$\left(\widetilde{e}_q^*\mathbf{M}(\mathbf{a})\right)_{\mathbf{k}} = \left(\mathbf{M}(\mathbf{a})\right)_{\mathbf{k}} \quad \text{for all } q \in p + l\mathbb{Z} \setminus L(\mathbf{k}^c, p).$$

Fix $q \in p + l\mathbb{Z} \setminus L(\mathbf{k}^c, p) = \{q \in p + l\mathbb{Z} \mid q \in \mathbf{k} \text{ or } q + 1 \notin \mathbf{k}\}$. Then, there exists a finite interval J such that $\left(\widetilde{e}_q^*\mathbf{M}(\mathbf{a})\right)_{\mathbf{k}} = \left(\widetilde{e}_q^*\left(\mathbf{M}(\mathbf{a})_J\right)\right)_{\mathrm{res}_J(\mathbf{k})}$. By Corollary 2.3.5, the right-hand side must be equal to $\left(\mathbf{M}(\mathbf{a})_J\right)_{\mathrm{res}_J(\mathbf{k})} = \left(\mathbf{M}(\mathbf{a})\right)_{\mathbf{k}}$. This proves the proposition. □

Now we can state the main result of this paper.

Theorem 6.4.3. *The image of $\mathcal{B}_{l-1}^{(1),ap}$ under the morphism $\Phi_{\mathbb{Z}}$ coincides with $(\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma(\mathbf{O}^*)$. In other words, $\Phi_{\mathbb{Z}} : \left(\mathcal{B}_{l-1}^{(1),ap}; \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*\right) \to \left((\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma(\mathbf{O}^*); \mathrm{wt}, \widehat{\varepsilon}_p^*, \widehat{\varphi}_p^*, \widehat{e}_p^*, \widehat{f}_p^*\right)$ gives an isomorphism of crystals.*

Proof. By Theorem 5.3.6 and Proposition 6.4.2, $\Phi_{\mathbb{Z}}(\mathcal{B}_{l-1}^{(1),ap})$ is a subcrystal of $(\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma$, and its crystal graph is connected. Also, $\Phi_{\mathbb{Z}}(\mathbf{a_0}) = \mathbf{O}^*$. Therefore, $\Phi_{\mathbb{Z}}(\mathcal{B}_{l-1}^{(1),ap})$ coincides with $(\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma(\mathbf{O}^*)$. Namely, $\Phi_{\mathbb{Z}}$ is a surjective map which preserves weights. More specifically, for $\widehat{\lambda} \in \widehat{Q}_- := \oplus_{p=0}^{l-1}\mathbb{Z}_{\le 0}\widehat{\alpha}_p$, set $\left(\mathcal{B}_{l-1}^{(1),ap}\right)_{\widehat{\lambda}} := \{\mathbf{a} \in \mathcal{B}_{l-1}^{(1),ap} \mid \mathrm{wt}(\mathbf{a}) = \widehat{\lambda}\}$ and $(\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma(\mathbf{O}^*)_{\widehat{\lambda}} := \{\mathbf{M} \in (\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma(\mathbf{O}^*) \mid \mathrm{wt}(\mathbf{M}) = \widehat{\lambda}\}$, respectively. Then, the restriction $\Phi_{\mathbb{Z}}|_{\widehat{\lambda}} : \left(\mathcal{B}_{l-1}^{(1),ap}\right)_{\widehat{\lambda}} \to (\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma(\mathbf{O}^*)_{\widehat{\lambda}}$ is a surjective map for each $\widehat{\lambda} \in \widehat{Q}_-$. Since both $\mathcal{B}_{l-1}^{(1),ap}$ and $(\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma(\mathbf{O}^*)$ are isomorphic to $B(\infty)$, the two sets $\left(\mathcal{B}_{l-1}^{(1),ap}\right)_{\widehat{\lambda}}$ and $(\mathcal{B}Z_{\mathbb{Z}}^e)^\sigma(\mathbf{O}^*)_{\widehat{\lambda}}$ are of the same (finite) cardinality. Consequently, $\Phi_{\mathbb{Z}}$ must be a bijection. This proves the theorem. □

The following is an easy consequence of Theorem 6.4.3.

Corollary 6.4.4. *The composite map* $* \circ \Phi_{\mathbb{Z}} : \mathcal{B}_{l-1}^{(1),ap} \xrightarrow{\Phi_{\mathbb{Z}}} (\mathcal{B}Z_{\mathbb{Z}}^{e})^{\sigma}(\mathbf{O}^{*}) \xrightarrow{*} \mathcal{B}Z_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ *gives rise to an isomorphism of crystals*

$$\left(\mathcal{B}_{l-1}^{(1),ap}; \mathrm{wt}, \widehat{\varepsilon}_p^{*}, \widehat{\varphi}_p^{*}, \widehat{e}_p^{*}, \widehat{f}_p^{*}\right) \xrightarrow{\sim} \left(\mathcal{B}Z_{\mathbb{Z}}^{\sigma}(\mathbf{O}); \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p\right).$$

Appendix A. Another explicit description

There is another explicit description of $\mathcal{B}Z_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ in terms of another crystal structure on $\mathcal{B}_{l-1}^{(1),ap}$; namely $\left(\mathcal{B}_{l-1}^{(1),ap}; \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p\right)$. In the appendix, we explain it. Because all results in this appendix are obtained by methods similar to the ones which we explained in this paper, we omit the details.

A.1. Let $I = [n+1, n+m]$ be a finite interval in $\mathbb{Z}$, and $\mathbf{k} = \{k_{n+1} < \cdots < k_{n+u}\} \in \mathcal{M}_I^{\times}$. For a given Lusztig datum $\mathbf{a}^I = (a_{i,j}^I)_{(i,j)\in\Delta_I^+} \in \mathcal{B}_I$ associated to the finite interval I, we introduce a new collection $\mathbf{M}'(\mathbf{a}^I) = (M'_{\mathbf{k}}(\mathbf{a}^I))_{\mathbf{k}\in\mathcal{M}_I^{\times}}$ of integers by

$$M'_{\mathbf{k}}(\mathbf{a}^I) := -\sum_{i=n+1}^{n+u}\sum_{j=k_i+1}^{n+m+1} a_{k_i,j}^I + \min\left\{ \sum_{n+1\le p<q\le n+u} a_{c_{p,q},c_{p,q}+(q-p)}^I \;\middle|\; \begin{array}{c} C=(c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau.}\end{array} \right\}.$$

Define a map Φ'_I by $\mathbf{a}^I \mapsto \mathbf{M}'(\mathbf{a}^I)$ (*cf.* (2.4.1)).

By using the map Φ'_I, we can construct an isomorphism from $\mathcal{B}_I$ to $\mathcal{B}Z_I$ directly (*cf.* Corollary 2.4.4).

Claim A.1. *For each* $\mathbf{a}^I \in \mathcal{B}_I$, $\Phi'_I(\mathbf{a}^I) = \mathbf{M}'(\mathbf{a}^I)$ *is an element of* $\mathcal{B}Z_I$. *Moreover, the map* $\Phi'_I : \mathcal{B}_I \to \mathcal{B}Z_I$ *is a bijection which gives rise to an isomorphism of crystals*

$$\left(\mathcal{B}_I; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right) \xrightarrow{\Phi'_I} \left(\mathcal{B}Z_I; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right).$$

A.2. We discuss the infinite case. Let $\mathbf{a} = (a_{i,j})_{(i,j)\in\Delta_{\mathbb{Z}}^+} \in \mathcal{B}_{\mathbb{Z}}$, and set $\mathbf{a}^I = (a_{i,j})_{(i,j)\in\Delta_I^+}$. It is obvious that $\mathbf{a}^I$ is an element of $\mathcal{B}_I$. We can prove the following statements:

Claim A.2. (1) *For* $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}$ *and* $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}^{c}$, *there exists a finite interval* I_0 *such that for every finite interval* $J \supset I_0$, $M'_{\mathrm{res}_J^c(\mathbf{k})}(\mathbf{a}^J) = M'_{\mathrm{res}_{I_0}^c(\mathbf{k})}(\mathbf{a}^{I_0})$.

(2) *Define* $M'_{\mathbf{k}}(\mathbf{a}) := M'_{\mathrm{res}_{I_0}^c}(\mathbf{k})(\mathbf{a}^{I_0})$. *Then, a collection* $\mathbf{M}'(\mathbf{a}) := (M'_{\mathbf{k}}(\mathbf{a}))_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}^{c}}$ *is an element of* $\mathcal{B}Z_{\mathbb{Z}}$. *In other words, we have a map* $\Phi'_{\mathbb{Z}} : \mathcal{B}_{\mathbb{Z}} \to \mathcal{B}Z_{\mathbb{Z}}$ *defined by* $\mathbf{a} \mapsto \mathbf{M}'(\mathbf{a})$.

(3) *The restriction of* $\Phi'_{\mathbb{Z}}$ *to* $\mathcal{B}_{l-1}^{(1),ap}$ *gives a bijection from* $\mathcal{B}_{l-1}^{(1),ap}$ *to* $\mathcal{B}Z_{\mathbb{Z}}^{\sigma}(\mathbf{O})$. *Moreover, it gives rise to an isomorphism of crystals*

$$\left(\mathcal{B}_{l-1}^{(1),ap}; \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p\right) \xrightarrow{\Phi'_{\mathbb{Z}}} \left(\mathcal{B}Z_{\mathbb{Z}}^{\sigma}(\mathbf{O}); \mathrm{wt}, \widehat{\varepsilon}_p, \widehat{\varphi}_p, \widehat{e}_p, \widehat{f}_p\right).$$

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